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1. Express $\det(\mathrm{adj}(A))$ in terms of $\det A$, where A is an $n \times n$-matrix.

Since $A\,\mathrm{adj}(A) = (\det A)I$, we have: $(\det A)\det(\mathrm{adj}(A)) = (\det A)^n$. Thus, when $\det A \neq 0$, $\det(\mathrm{adj}(A)) = (\det A)^{n-1}$. The same remains true when $\det A = 0$. This is obvious when $A = 0$ since then $\mathrm{adj}(A) = 0$. When $A \neq 0$ but $\det A = 0$, we still have $A\,\mathrm{adj}(A) = 0$, which is possible only if $\mathrm{adj}(A)$ is degenerate.

2. Solve system of linear equations:

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 - 4x_4 &=& 4 \\ x_2 - x_3 + x_4 &=& -3 \\ x_1 + 3x_2 \qquad - 3x_4 &=& 1 \\ -7x_2 + 3x_3 + x_4 &=& -3 \end{array}.$$

Following the row reduction algorithm, we subtract the 1st equation from the 3rd one, then subtract the 2nd equation 5 times from the 3rd one, add it 7 times to the 4th one, and finally add what has become the 3rd equation twice to the 4th one. We arrive at the following system:

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 - 4x_4 &=& 4 \\ x_2 - x_3 + x_4 &=& -3 \\ 2x_3 - 4x_4 &=& 12 \\ 0 &=& 0 \end{array}.$$

Denoting by t the value of x_4, which can be arbitrary, and performing the back substitution, we find:

$$x_4 = t,\ x_3 = 6 + 2t,\ x_2 = 3 + t,\ x_1 = -8.$$

3. Use Sylvester's rule to find inertia indices of quadratic form:

$$x_1x_2 - x_2^2 + x_3^2 + 2x_2x_4 + x_4^2.$$

It is convenient, before applying Sylvester's rule, to reorder the variables: $x_1 = y_4,\ x_2 = y_3,\ x_3 = y_2,\ x_4 = y_1$. Then, in the y-coordinates, the quadratic form has coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Computing the leading minors, we find:

$$\Delta_0 := 1,\ \Delta_1 = 1,\ \Delta_2 = 1,\ \Delta_3 = -2,\ \Delta_4 = -1/4.$$

In the sequence $+.+,+,-,-$ of their signs, there is one sign change, and therefore the negaive and positiove inertia indices of this quadratic form are $q = 1$ and $p = 4 - q = 3$.

4. Transform quadratic form $x_1x_2 + x_3x_4$ to the normal form by an orthogonal transformation.

We apply 45°-rotations of the coordinate system in each (x_1, x_2)-plane and in (x_3, x_4)-plane:

$$x_1 = \frac{y_1 - y_2}{\sqrt{2}},\ x_2 = \frac{y_1 + y_2}{\sqrt{2}},\ x_3 = \frac{y_3 - y_4}{\sqrt{2}},\ x_4 = \frac{y_3 + y_4}{\sqrt{2}}.$$

In the new coordinates, the quadrtic form is $y_1^2/2 - y_2/2 + y_3^2/2 - y_4^2/2$. Switching y_2 and y_3 we obtain the standard normal form of the sum of the squares of coordinates with coefficients $1/2, 1/2, -1/2, -1/2$ ordered non-increasingly.

5. Find the Jordan normal form of matrix:

$$\begin{bmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{bmatrix}.$$

Denote the matrix by A. First, we compute the characteristic polynomial $\det(\lambda I - A)$, and find: $\lambda^3 + 2\lambda^2 + \lambda = \lambda(\lambda+1)^2$. Therefore A has the eigenspace of dimension 1 corresponding to the eigenvalue $\lambda = 0$, and the root space of dimension 2 corresponding to $\lambda = -1$. Next, we look for the dimension of the the eigenspace corresponding to eigenvalue $\lambda = -1$. For this, we examine the matrix $A + I$:

$$\begin{bmatrix} 1 & 3 & 3 \\ -1 & 9 & 6 \\ 2 & -14 & -9 \end{bmatrix}.$$

It has rank 2 (since $\begin{vmatrix} 1 & 3 \\ -1 & 9 \end{vmatrix} = 12 \neq 0$). Therefore the eigenspace corresponding to $\lambda = -1$ has dimension 1 while the root space has dimension 2. Thus the Jordan normal form of the matrix is:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

6. Can a non-zero anti-symmetric matrix be nilpotent? If "yes" give an example, if "no" provide a proof.

`Answer:` *No.*

`Proof:` According to the Spectral Theorem, every anti-symmetric matrix is diagonalizable over $\mathbb{C}$. To be nilpotent, a diagonal matrix must be zero, in which case a matrix similar to it is zero too.

7. Classify all linear operators in $\mathbb{R}^2$ up to linear changes of coordinates.

When the characteristic polynomial has complex conjugate roots $a \pm bi$ (where we may assume $b > 0$), the matrix is similar to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. When the characteristic polynomial has two distinct real roots $\lambda_1 > \lambda_2$, the matrix is similar to the diagonal one: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. In the remaining case when the characteristic polynomial has double root λ_0, the matrix is similar to scalar matrix $\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$ or to Jordan cell $\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$ depending on whether the eigenspace has dimension 2 or 1.

8. Find all those values of $a_1, \dots, a_n$ for which the following matrix is nilpotent:

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & . \\ 0 & \dots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \end{bmatrix}.$$

Using the cofactor expansion with respect to the 1st column and applying induction on n, it is easy to chow that the characteristic polynomial of the matrix is equal to $\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n$. As it follows from the Jordan Canonical Form Theorem, a matrix is nilpotent if and only if all complex roots of the characteristic polynomial are equal to zero. Therefore our matrix is nilpotent if and only if $a_1 = \dots = a_n = 0$.

9. Find out if the following quadratic hypersurfaces in $\mathbb{C}^4$ can be transformed into each other by linear ingomogeneous changes of coordinates:

$$z_1z_2 + z_2z_3 + z_3z_4 = 1 \quad \text{and} \quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = z_1 + z_2 + z_3 + z_4.$$

Quadratic form $z_1z_2 + z_2z_3 + z_3z_4$ from the first equation has coefficient matrix

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The determinant of this matrix (as it is easy to see) is non-zero, and hence the quadratic form is equivalent over $\mathbb{C}$ to $z_1^2 + z_2^2 + z_3^2 + z_4^2$.

Completing squares in the second equation, we transform it to the form

$$(z_1 - \frac{1}{2})^2 + (z_2 - \frac{1}{2})^2 + (z_3 - \frac{1}{2})^2 + (z_4 - \frac{1}{2})^2 = 1.$$

Thus, both hypersurfaces are equivalen to the complex sphere

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1.$$

10. Prove that any orthogonal transformation in $\mathbb{R}^4$ with the determinant equal to -1 has an invariant 3-dimensional subspace.

As it follows from the real version of the Spectral Theorem applied to orthogonal transformations, each orthogonal transformation in $\mathbb{R}^4$ with determinant -1 in a suitable orthonormal basis is described by matrix:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

In particular, it has real eigenvectors with the eigenvalues -1 (the last column) and 1 (next to last). The 3-dimensional subspace perpendicular to either of these eigenvectors is invariant.