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A Useful Principle in Solving Differential Equations

1. THE IDEA.

Consider a first-order differential equation

(1) $$y' = F(x, y).$$

As we have seen, this is in general satisfied, not just by one function $y = f(x)$, but by a *family* of functions. A case of this phenomenon that we are long familiar with is when (1) has the form

(2) $$y' = F(x).$$

Then we know that the family of solutions has the form

(3) $$y = G(x) + C,$$

where $G(x)$ is any antiderivative of $F(x)$, and C ranges over all real numbers. Since we have learned many techniques of integration (finding antiderivatives), it is useful to have techniques that reduce the solution of other sorts of differential equations to that case. One situation where we have seen that we can do so is when the given equation is separable.

Another very wide class of cases is based on paying attention to how the solutions of our given equation relate to one another. Here is the general principle.

(4) *Given a differential equation* (1), *examine the way different solutions must be related, and look for a* ***change of variables*** *that will turn the solutions of* (1) *into a family of functions related to one another simply by addition of constants, as in* (3). *After this change of variables, the differential equation* (1) *will reduce to one of the form* (2), *and hence can be solved by integration.*

In the next two sections, we will develop a standard technique for solving *first-order linear* differential equations, using the above principle as one of the paths leading to that technique.

2. FIRST ORDER HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.

A first-order differential equation (1) is said to be **linear** if $F(x, y)$ is linear as a function of y; in other words, if the equation has the form

(5) $$y' = P(x)\,y + Q(x).$$

Note that in this equation, the terms y' and $P(x)\,y$ are of degree 1 in y and its derivative, while $Q(x)$ is of degree 0 in those variables. An equation is called **homogeneous** in a set of variables if all the terms have the same degree in those variables. Thus, (5) is homogeneous of degree 1 in y and its derivative only if $Q(x) = 0$; i.e., if the equation has the form

(6) $$y' = P(x)\,y.$$

In this section we will see how to solve homogeneous equations (6). In the next section we will climb on the shoulders of that result, and solve nonhomogeneous equations (5).

I claim that the solutions to (6) are

(7) $$y = c\,e^{\int P(x)\,dx},$$

for real numbers c. That is, if A is any antiderivative of $P(x)$, then the solutions to (6) are the functions

(8) $$y = c\,e^{A(x)}.$$

It is easy to check that these functions do indeed satisfy (6). But how could we have *discovered* this solution? Here are three independent approaches, each of which gives us a different bit of insight into differential equations generally.

Motivation #1. Our general principle (4) says that we should look at how different solutions to (6) are related to one another. Clearly, if $y = f(x)$ is a solution to (6), then so is $y = c f(x)$ for every real constant c. Hence (4) says that we should look for a change of variables that will turn solutions related in this way into solutions related by an additive constant. *Multiplying* a function by a positive constant corresponds to *adding* a constant to its logarithm; so let us make the substitution

(9) $$u = \ln y; \quad \text{equivalently,} \quad y = e^{u}.$$

When we put this into (6), that differential equation becomes $u' e^{u} = P(x)\,e^{u}$, which simplifies to $u' = P(x)$. This is, as we hoped, a differential equation of the form (2), hence we can solve it by integration, getting $u = \int P(x)\,dx + C$, i.e., $A(x) + C$, for $A(x)$ an antiderivative of $P(x)$. Substituting this into the right-hand equation of (9), we get (8), where $c = e^{C}$.

Actually, this only gives the case of (8) where c is positive, since our substitution $u = \ln y$ only makes sense for positive-valued functions y. However, once we have found this family of solutions, it is easy to see that zero and negative values of c also give solutions to (6). So if we think of our general principle (4) simply as a guide, it has indeed guided us to the solution (8).

Assuming the domain of definition of our functions is an interval (possibly infinite), (8) in fact gives *all* solutions to (6). To see this, let $z = e^{A(x)}$ and let y be any other solution to (6). We want to show that y/z is constant. If we differentiate y/z, and use the fact that both y and z are solutions to (6), we get

$$\begin{aligned}(y/z)' &= (y'z - y z')/z^2 \\ &= ((P(x)\,y)\,z - y\,(P(x)\,z))/z^2 \\ &= 0/z^2 = 0.\end{aligned}$$

So y/z is indeed a constant c, so $y = c\,z = c\,e^{A(x)}$.

Let us now look at a different way we could have discovered the solution (8) of (6).

Motivation #2. Notice that the equation (6) is *separable*, as defined on p. 594 of our text. Hence following the method shown there, we divide by the function of y appearing on the right-hand side of (6), in this case y itself, getting

(10) $$y'/y = P(x),$$

and integrate both sides of this equation, getting

(11) $$\ln y = \int P(x)\,dx + C.$$

To get y we exponentiate, which again gives (7). (The same comments about negative c and uniqueness given in Motivation #1 apply here.)

Here is a brief sketch of a third avenue leading to (8):

Motivation #3. Assume we want to solve (6) on some interval $[a, b]$. We may subdivide $[a, b]$ using points $a = x_0 < x_1 < \ldots < x_n = b$, and think of $P(x)$ as approximately constant on each interval $[x_{i-1}, x_i]$, with some constant value $P(x_i^*)$ $(x_{i-1} \le x_i^* \le x_i)$. Then on that interval, (6) can be approximated by the equation $y' = P(x_i^*)\,y$, for which we know the general solution is any constant times the exponential function $e^{P(x_i^*)\,x}$ (§3.8 of Stewart; also pp. 580 and 606 [2]). Hence, as x moves from x_{i-1} to x_i, the value of y will be multiplied by approximately $e^{P(x_i^*)\,\Delta x_i}$, where $\Delta x_i = x_i - x_{i-1}$. So as x goes all the way from a to b, y is multiplied by approximately

$$e^{P(x_1^*)\,\Delta x_1} \cdot e^{P(x_2^*)\,\Delta x_2} \cdot \ldots \cdot e^{P(x_n^*)\,\Delta x_n}$$
$$= e^{\Sigma_i P(x_i^*)\,\Delta x_i}.$$

As our subdivision of $[a, b]$ becomes finer and finer, this should approach $e^{\int_a^b P(t)\,dt}$. Replacing the upper index of integration by the variable x to get a function, we see that it will have the form (7).

The above discussion is too sketchy to be a proof; but again, it guides us to the solution (7), which we can then verify by substituting it into the given equation.

Here are some examples of this method.

Example 1. Solve $y' = y \sin x$. Find the particular solution for which $y = 1$ when $x = 0$.

Solution. An antiderivative of $\sin x$ is $-\cos x$, so (7) gives the general solution $y = c\,e^{-\cos x}$.

To find the particular solution with $y = 1$ when $x = 0$, we substitute $x = 0$, $y = 1$, getting $1 = c\,e^{-\cos 0} = c\,e^{-1}$. So $c = e$, so the particular solution is $y = e \cdot e^{-\cos x} = e^{1 - \cos x}$.

Example 2. Solve $y' = -y x$.

Solution. An antiderivative of $-x$ is $-x^2/2$, so the expression (7) gives $y = c\,e^{-x^2/2}$.

Example 3. Solve $y' = r y / x$, where r is any real number.

Solution. An antiderivative of r/x is $r \ln |x|$, so (7) gives $y = c\,e^{r \ln |x|} = c\,|x^r|$.

3. THE NONHOMOGENEOUS CASE.

We are now ready to tackle the more general equation (5),

$$y' = P(x)\,y + Q(x).$$

Our general principle (4) says that we should look at the relationship among solutions to this equation. So suppose y_1 and y_2 are two solutions to (5); i.e., that

$$y_1' = P(x)\,y_1 + Q(x),$$
$$y_2' = P(x)\,y_2 + Q(x).$$

If we subtract these equations, we get

$$y_1' - y_2' = P(x)(y_1 - y_2).$$

This says that $y_1 - y_2$ is a solution to the *homogeneous* equation (6),

$$y' = P(x)\,y,$$

which we learned how to solve in the preceding section. It is not hard to check by turning the above calculation backward that, conversely, if we add to any solution to our nonhomogeneous equation (5) a solution to the homogeneous equation (6), we again get a solution to (5).

So if we let $g(x)$ be any nonzero solution to (6), i.e., any nonzero function satisfying

(12) $$g'(x) = P(x)\,g(x),$$

then the solutions to (5) will differ among themselves only by constant multiples of $g(x)$. The principle (4) now tells us to make a change of variables that will transform these into functions that differ among themselves by addition of *constants*. To do this we should clearly divide by $g(x)$; i.e., let

(13) $$u = y/g(x), \quad \text{equivalently,} \quad y = u\,g(x).$$

Principle (4) says that this substitution should change (5) into a differential equation that can be solved by integration. To see whether this works, let us substitute (13) into (5), and simplify.

$$(u\,g(x))' = P(x)\,(u\,g(x)) + Q(x)$$

$$u'g(x) + u\,g(x)P(x) = P(x)\,u\,g(x) + Q(x) \qquad \text{(by (12))}$$

$$u'g(x) = Q(x)$$

$$u' = Q(x)/g(x).$$

So we can indeed now find u by integrating $Q(x)/g(x)$. We then get y by multiplying u by $g(x)$ (see (13)). In summary,

(14) *To solve a nonhomogeneous linear differential equation* (6)*, first find a nonzero solution* $g(x)$ *to the corresponding homogeneous linear differential equation* (7)*. Then make the substitution* $y = u\,g(x)$ *in* (6)*, getting an equation that can be solved by integration, and substitute back to obtain* y*.*

The resulting family of solutions is described by

$$y = g(x)\,(\textstyle\int (Q(x)/g(x))\,dx + C).$$

You might prefer either to learn the substitution used in the above procedure, or to memorize the final formula, or both. If you memorize the formula, be careful to remember that $g(x)$ denotes a solution to the corresponding homogeneous equation.

Example 4. Solve $y' = xy + x^3$.

Solution. We must first find a nonzero solution $g(x)$ to the corresponding homogeneous equation, i.e., a function satisfying $g' = xg$. The method of §2 tells us that such a solution is $e^{\int x\,dx} = e^{x^2/2}$. (Since we only need one such solution, we have left out the constant ‘‘c’’ of the formula in that section.) The formula at the end of (14) now gives

(15) $$y = e^{x^2/2}\left(\int (x^3/e^{x^2/2})\,dx + C\right).$$

Writing the integral as $\int x^3 e^{-x^2/2}\,dx$, we make the substitution $u = -x^2/2$, getting $\int 2ue^u\,du$. Integration by parts gives $2ue^u - 2e^u = 2(u-1)e^u$. Expressing this in terms of x, we find that our integral equals $2(-x^2/2-1)e^{-x^2/2} = -(x^2+2)e^{-x^2/2}$. Thus, (15) gives

$$\begin{aligned} y &= e^{x^2/2}\left(-(x^2+2)e^{-x^2/2} + C\right) \\ &= -x^2 - 2 + Ce^{x^2/2}. \end{aligned}$$

After all this computation, it is worth checking that these functions do indeed satisfy the given differential equation. You will not find it hard do so.

Remark: In Stewart and many other texts, first-order linear differential equations are written with the $P(x)y$ on the left side of the equation instead of on the right as in (5) and (6):

(16) $$y' + P(x)y = Q(x) \qquad \text{(nonhomogeneous)},$$

(17) $$y' + P(x)y = 0 \qquad \text{(homogeneous)}.$$

This is because they will eventually be looked at in the context of higher order equations,

(18) $$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_0(x)y = Q(x).$$

I used the forms (5) and (6) because I wanted to discuss these equations as instances of (1). There is no essential difference in the method of solution (and in real life, such equations are at least as likely to appear in this form as the other). But note that because $P(x)$ effectively has opposite signs in the two formulations, the expression (7) $e^{\int P(x)\,dx}$ of our development corresponds to its inverse in Stewart's development; so corresponding to the final formula of our solution (14), Stewart gets a formula in which one *multiplies* inside the integral and *divides* outside the integral by $e^{\int P(x)\,dx}$.

4. ANOTHER APPLICATION OF OUR GENERAL PRINCIPLE.

In the two cases with which we have illustrated the general principle (4) above, the change of variables that we made replaced the dependent variable y by a new dependent variable u, but kept the independent variable x unchanged. The method is not limited to such cases. I give below a nice class of examples of a different sort.

This section is not required reading for Math (H)1B; in fact, the method it leads to is part of the curriculum of Math 54. But since (4) is an important tool in the theory of differential equations, it is instructive to see a variety of applications.

Suppose a differential equation has the form

(19) $$y' = F(y/x).$$

Notice that this means that the slope of the direction field is constant along all lines through the origin: on each line $y = cx$, the slope shown by the direction field is everywhere $F(c)$.

It is not hard to see from this that the operation of expanding or shrinking the picture of any solution-curve by a fixed nonzero factor r (i.e., multiplying both coordinates of every point by the same constant r) will give the picture of another solution-curve. That is, if the curve $y = f(x)$ is one solution to the equation, then for any nonzero r, another solution will be the curve $y/r = f(x/r)$; in other words

$y = r f(x/r)$.

How do we make a change of variables that will turn this system of curves into curves that differ by additive constants?

Well, the process of ''expanding or shrinking'' a solution as described above takes a curve containing a point (x, y) to a curve contain the point (rx, ry). So let us start by going to coordinates in which (x, y) and (rx, ry) have the same value of the independent variable. We can achieve this by letting our new independent variable be

(20) $$u = y/x.$$

We can then take x as our new dependent variable, and eliminate y, using (20) in the form $y = ux$. To express (19) in terms of our new variables, it is easiest to pass to differential notation:

(21)
$$\begin{aligned} dy/dx &= F(y/x) \\ dy &= F(u)\,dx \\ d(ux) &= F(u)\,dx \\ u\,dx + x\,du &= F(u)\,dx \\ x\,du &= (F(u) - u)\,dx \\ dx/du &= x/(F(u) - u). \end{aligned}$$

The original ''expanding or shrinking'' operation, which leaves our new independent variable u unchanged, still multiplies our new dependent variable x by the constant r; so (as in §2) we can convert it into addition of a constant by taking logarithms. Thus, let us pass to a new dependent variable $v = \ln x$, i.e., let $x = e^v$. The last line of (21) then becomes $(dv/du)e^v = e^v/(F(u) - u)$, i.e., $dv/du = 1/(F(u) - u)$. This we can at last solve by integration, getting $v = \int du/(F(u) - u) + C$. We then substitute back, to get an equation relating x and y.

The above discussion points us to a somewhat more detailed description of the procedure (4) for cases where our system of transformations do not, initially, preserve the independent variable:

(22) *Given a differential equation* (1), *find a family of transformations of the plane which carries solution-curves of* (1) *to other solutions-curves of* (1). *Then make a change of variables so that all these transformations preserve the new independent variable. Finally, choose a new dependent variable so that the transformations are given by addition of a constant to that variable. The differential equation* (1) *will then reduce to one that can be solved by integration.*

The transformations we have discussed in this note, that carry solution-curves to solution-curves, are called by specialists in the theory of differential equations *symmetries* of the given equation. (A Google Book search for ''symmetries'' together with ''differential equations'' gives, at the moment, 791 results, for your reading pleasure.)

The ideas of this handout are also used in connection with higher-order differential equations, where families of symmetries can be used to reduce higher-order differential equations to a combination of integrations and the solving of lower-order differential equations.