ON JACOBSON RADICALS OF GRADED RINGS

by George M. Bergman

Rings are associative with 1. In §§1-2 we shall prove the following results, thus getting the well-known results Corollaries 3 and 4 via more general arguments than those I have seen in the literature:

<u>Theorem 1.</u> Let R be a $(\mathbb{Z}/n\mathbb{Z})$-graded ring, with homogeneous component of degree $i \in \mathbb{Z}/n\mathbb{Z}$ written R_i. Then if $a_0+\ldots+a_{n-1} \in J(R)$ $(a_i \in R_i)$, we have $na_i \in J(R)$ $(0 \leq i < n)$.

<u>Corollary 2.</u>* If R is a $\mathbb{Z}$-graded ring, then $J(R)$ is a homogeneous ideal.

<u>Corollary 3.</u> (reference?) If R is a ring, and $R[t]$ the extension of R by a central indeterminate t, then $J(R[t]) = I[t]$, where $I = J(R[t]) \cap R$, a nil ideal of R.

<u>Corollary 4.</u> (Amitsur. See [1, p.252]). Let R be an algebra over a field F, and $F(t)$ the field of rational functions over F, in an indeterminate t. Then $J(R \otimes_F F(t)) = I \otimes_F F(t)$, where $I = J(R \otimes_F F(t)) \cap R$, a nil ideal of R.

Some generalizations of these results are also indicated.

The radicals referred to above are defined without reference to the grading on R. However (as Hochschild reminded me, though it was already secretly nagging at my conscience when I was writing sections 1 and 2) there is a version of the concept of the Jacobson radical appropriate to graded rings per se, and in §3 we shall see that the same methods used in §§1-2 yield information on the relation between the graded and ungraded radicals of R. In particular, when the grading group G is $\mathbb{Z}$ (or more generally, is torsion-free abelian) we get $J(R) \subseteq J_G(R)$, while when G is finite, it appears likely that one has the opposite inclusion, $J_G(R) \subseteq J(R)$! I prove the latter inclusion for G finite solvable. Possible attacks on the general case are discussed in §§4-5.

[Rough draft. Comments and related references welcomed. Group theorists note §§3-5.]

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Passman, <u>Infinite crossed products</u>, Proof of Theorem 22.6. (But in statement of that Theorem, "nilpotent" should be "nil".)

1. Simple modules and finite extensions. We need three preliminary Lemmas, of some interest in themselves. Throughout this section, $R \subseteq S$ will be rings such that S is finitely generated as an R-module, by R-centralizing elements:

$$S = R\,a_1 + \ldots + R\,a_n, \qquad [R, a_i] = 0.$$

Lemma 5. Let M be a simple left S-module. Then M is semisimple of finite length as a left R-module, and all its simple R-submodules are isomorphic.

Proof. For any nonzero $x \in M$, we have

$M = Sx = Ra_1x + \ldots + Ra_nx$, so M is finitely generated as an R-module. Hence it has a maximal R-submodule N. For $i = 1, \ldots, n$, let $N_i \subseteq M$ denote $\{y \in M \mid a_iy \in N\}$. Because each a_i commutes with R, multiplication by a_i is an R-module endomorphism of M, hence each N_i, being the inverse image of a maximal submodule under a module homomorphism, is either a maximal submodule of M, or all of M. Letting $L = N_1 \cap \ldots \cap N_n$, we see that M/L is an R-module of finite length. But note that $S\,L = \sum R\,a_i\,L \subseteq \sum R\,a_i\,N_i \subseteq \sum R\,N = N \neq M$, so as M was a simple S-module, $L = 0$, hence the R-module M has finite length.

Hence we can choose a simple R-submodule $M_0 \subseteq M$. Now $M = S\,M_0 = \sum a_i\,M_0$, a sum of finitely many homomorphic images of the simple module M_0, from which the desired conclusion follows by the general properties of semisimple modules. ||

Lemma 6. Let M be a simple R-module. Then $S \otimes_R M$ is semisimple of finite length as an R-module, a direct sum of copies of M.

Proof. Clearly, $S \otimes_R M$ is spanned as an R-module by the submodules $a_i \otimes M$, each of which is a homomorphic image of M. ||

Lemma 7. $J(S) \cap R \supseteq J(R)$, with equality if $S \otimes M$ is nonzero for every simple R-module M, or if R is rationally closed in S (i.e., $x \in R$, $x^{-1} \in S \Rightarrow x^{-1} \in R$.)

Proof. Say $r \in R - J(S)$. Then we can find a simple S-module M not annihilated by r. As an R-module, M will be semisimple by Lemma 5, hence will have a simple R-submodule M_0 not annihilated by r. This shows $r \notin J(R)$, giving the first inclusion.

Next, let $r \in R - J(R)$. Suppose $S\otimes-$ kills no simple R-modules. Then we take a simple R-module M not annihilated by r, and form $S \otimes_R M$. This will be a nonzero cyclic S-module, hence have a simple factor-module $M' = (M \otimes_R S)/N$. From Lemma 6 we see that as an R-module, M' is a direct sum of copies of M, hence it is not annihilated by r, hence $r \notin J(S)$, as desired. Finally, under the same hypothesis $r \in R - J(R)$, suppose R is rationally closed in S. Since $r \notin J(R)$ there is an element $x \in RrR$ such that $1 - x$ is not invertible in R. Hence $1 - x$ is not invertible in S, hence $x \notin J(S)$. ‖

Open question: Is the inclusion of Lemma 7 ever strict?

2. Proofs of the main results, and generalizations.

Proof of Theorem 1. Let $\mathbb{Z}[\omega]$ be the extension of $\mathbb{Z}$ in the complex numbers by a primitive n^{th} root of unity ω. This is free as a $\mathbb{Z}$-module, of rank $\varphi(n)$. Thus, if we take $S = R \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$, $R \subseteq S$ will satisfy the hypothesis of the preceding section, and since S is free as a right R-module, $S\otimes-$ will not annihilate any nonzero R-modules. Hence by Lemma 7, $J(S) \cap R = J(R)$. In particular, $a_0 + \ldots + a_{n-1} \in J(S)$.

Now we may grade S by putting $S_i = R_i \otimes \mathbb{Z}[\omega]$, and we may then define an automorphism σ on S by $\sigma(\sum s_i) = \sum \omega^i s_i$ $(s_i \in S_i)$. Clearly, an automorphism sends $J(S)$ into itself. Now for any $i \in \mathbb{Z}/n\mathbb{Z}$ we compute: $\sum_j \omega^{-ij} \sigma^j(a_0 + \ldots + a_{n-1}) = n\, a_i$, hence this element lies in $J(S) \cap R = J(R)$, as claimed. ‖

Proof of Corollary 2. Say $a_r + \ldots + a_s \in J(R)$, where $r \leq s \in \mathbb{Z}$, $a_i \in R_i$. Now the $\mathbb{Z}$-grading on R induces a $\mathbb{Z}/n\mathbb{Z}$-grading for each positive integer n. If we take $n > r-s$, the $\mathbb{Z}/n\mathbb{Z}$-homogeneous components of our element will be precisely the a_i's (and 0), hence for each i, $n\, a_i \in J(R)$. But similarly $(n+1)a_i \in J(R)$, hence $a_i \in J(R)$. ‖

Proof of Corollary 3. Clearly $I[t]$ (the ideal of all elements of $R[t]$ with coefficients from I) is contained in $J(R[t])$. Conversely let $a_n t^n + \ldots + a_0 \in J(R[t])$. By the preceding Corollary each $a_i t^i$ lies in $J(R[t])$. But $R[t]$ has an automorphism sending t to $t+1$, hence $a_i(t+1)^i \in J(R[t])$. Now applying the same Corollary to this element to extract the degree-zero term, we get $a_i \in J(R[t])$. Hence $a_i \in I$. To show that I is nil, note that if $x \in I$, then $1 + xt$ will be invertible, and its inverse must have the form $1 - xt + x^2t^2 - \ldots$, hence for this to be a polynomial, some power of x must be zero. ||

Proof of Corollary 4. Again, clearly $I \otimes F(t) \subseteq J(R \otimes F(t))$. To get the converse, take an element of $J(R \otimes F(t))$, and write it $f(t)^{-1}(a_m t^m + \ldots + a_0)$ ($f \in F[t] - \{0\}$, $a_i \in R$). Then clearly $a_m t^m + \ldots + a_0 \in J(R \otimes F(t))$, and we need to prove that each $a_i \in J(R \otimes F(t))$. Now $F(t)$ is not $\mathbb{Z}$-graded, but it is $\mathbb{Z}/n\mathbb{Z}$-graded for each $n > 0$, with homogeneous components $F(t)_i = t^i F(t^n)$. If we apply Theorem 1 to the induced grading of $R \otimes F(t)$, we see that when $n > m$, we get $n\, a_i\, t^i \in J(R \otimes F(t))$, hence again taking two relatively prime values of n we have $a_i t^i \in J(R \otimes F(t))$, hence multiplying by t^{-i}, we have $a_i \in I$, as desired. To show that I is nil, consider $F(t)$ as embedded in the ring of formal Laurent series, $F((t))$. Then in $R \otimes F((t))$, we get the expansion $(1+xt)^{-1} = 1 - xt + x^2t^2 - \ldots$. But for this to lie in the indicated tensor-product ring, the coefficients of the powers of t, namely $1, -x, x^2, \ldots$ must lie in a finite-dimensional F-subspace of R; which means x must be algebraic over R. If the algebraic element x were not nilpotent, some polynomial in x with zero constant term would be a nonzero idempotent, contradicting $x \in J(R \otimes F(t))$. So x is nilpotent, as required. ||

It is easy to get many more results like Corollaries 3 and 4, e.g. by adjoining to R several noncommuting but R-centralizing indeterminates; by tensoring an algebra R over a commutative ring C with appropriate rings lying between C and the rational function field of its field of fractions; etc..

It would be interesting to know for what classes of groups G one has analogs of Theorem 1 or Corollary 2, with G in place of $\mathbb{Z}$ or $\mathbb{Z}/n$. Let us obtain one easy generalization of Corollary 2. We need two Lemmas:

Lemma 8. Suppose $R = \varinjlim R_\alpha$, the direct limit of a directed system of rings. Then $J(R) \supseteq \limsup J(R_\alpha)$, where the right-hand side is defined to consist of those $r \in R$ which are images of members of $J(R_\alpha)$ for some cofinal set of indices α. Equality holds if all the morphisms of our directed system carry nonunits to nonunits (e.g., are inclusions of rationally closed subrings.)

Proof. Straightforward using the characterization of $J(R)$ as $\{x \mid 1+RxR$ consists of units$\}$.$\|$

Lemma 9. Let $H \subseteq G$ be groups, R a G-graded ring, and $R_H = \sum_H R_h \subseteq R$, an H-graded subring. Then R_H is rationally closed in R.

Proof. Immediate by the homogeneity of $1 \in R$.$\|$

Theorem 10. Corollary 2 holds with $\mathbb{Z}$ replaced by any torsion-free abelian group.

Proof. Follows from Lemma 8, Lemma 9, Corollary 2, and the fact that any torsion-free abelian group is a direct limit of free abelian groups of finite ranks. To prove the result for $\mathbb{Z}^n$ one makes successive use of the n $\mathbb{Z}$-gradings.$\|$

One can similarly get Theorem 1 for any finite abelian group A (with $|A|$ for n) but I know nothing about the noncommutative case.

3. Graded radicals. If R is a G-graded ring, then along with $J(R)$ it is natural to look at the following ideal:

Definition. 11. Let G be a group, R a G-graded ring. Then by $J_G(R)$ we will denote the ideal which may be characterized equivalently as follows:

(1) The set of elements of R annihilating all simple G-graded R-modules.

(2) The largest homogeneous ideal $J \subseteq R$ such that $1+x$ is a unit for all $x \in J \cap R_e$ (e the identity element of G).

(3) The largest homogeneous ideal $J \subseteq R$ such that $J \cap R_e \subseteq J(R_e)$

If H is any subgroup of G, and R_H denotes the H-graded subring $\sum_H R_h \subseteq R$, then $J_G(R)$ can also be described inductively as

(4) The largest homogeneous ideal $J \subseteq R$ such that $J \cap R_H \subseteq J_H(R_H)$.

(Clearly (1) defines an ideal. The characterization of this ideal by (2) is proved exactly analogously to the corresponding result for ungraded rings. The equivalence of (2), (3), (4) is straightforward.)

From condition (2) above, we get

Lemma 12. $J_G(R)$ contains all homogeneous elements of $J(R)$. Hence by Theorems 10 and 1, when G is torsion-free abelian $J(R) \subseteq J_G(R)$, and when $G = \mathbb{Z}/n\mathbb{Z}$, $nJ(R) \subseteq J_G(R)$. ||

To see that the first of the above two inclusions can be strict, let C be an ungraded commutative domain and $C[t]$ its polynomial ring, $\mathbb{Z}$-graded in the standard way. Then $J(C[t]) = 0$, but $J_{\mathbb{Z}}(C[t]) = J(C) + tC[t]$ (easily seen from (3). If we adjoin an inverse to t, we again have $J(C[t,t^{-1}])=\{0\}$, and now $J_{\mathbb{Z}}(C[t,t^{-1}]) = J(C)[t,t^{-1}]$ which can again be nonzero.) For examples where the second inclusion is strict just take a case where $R = R_0$ so that $J_G(R) = J(R)$, but where $J(R) \neq nJ(R)$. To show the first inclusion (and Corollary 2) can fail when $G = \mathbb{Z}/n\mathbb{Z}$, take for n a prime p and for R the group algebra of G over a field of characteristic p. Then $J(R)$ is the augmentation ideal, but $J_G(R) = 0$.

But there is evidence that for G finite one has the opposite inclusion, $J_G(R) \subseteq J(R)$! I shall prove this for finite solvable G. We first need a Lemma allowing us to pass to composition factors of G:

Lemma 13. Let G be a group, N a normal subgroup, and R a G-graded ring. Suppose that $J_N(R_N) \subseteq J(R_N)$ and $J_{G/N}(R) \subseteq J(R)$. Then $J_G(R) \subseteq J(R)$. (Here $J_{G/N}(R)$ is the radical of R with respect to the weaker grading by G/N.)

Proof. By Def. 11, (4), $J_G(R) \cap R_N \subseteq J_N(R_N)$, and this ideal is $\subseteq J(R_N)$ by hypothesis. But R_N is the identity component of R under the G/N grading; hence $J_G(R)$ is a (G-homogeneous and thus) G/N-homogeneous ideal J whose intersection with the identity component of that grading is contained in the radical thereof. Hence by Def. 11, (3), $J_G(R)$ is contained in $J_{G/N}(R)$, which by hypothesis is contained in $J(R)$. ||

The next thing we need is some observations on graded modules. If R is a G-graded ring, we clearly have a "forgetful functor" from G-graded R-modules to ungraded R-modules. This functor has a right adjoint, which associates to an ungraded R-module M the graded R-module $M^G = \oplus_G M^g$, where each M^g is a copy of M, written $\{x^g \mid x \in M\}$, and M^G is made an R-module by defining, for $r \in R_h$, $r \cdot x^g = (rx)^{hg}$.* We remark that if we apply the forgetful functor to get an ungraded R-module from M^G, this will not in general be a direct sum of copies of M — the component M^g's are not R-submodules (though they are R_e-submodules). In fact, if M is simple and G is finite, I don't even know whether M^G will be of finite length, which obviously makes for difficulties in using this construction in studying graded and ungraded radicals of R. However, we

*To get the full category-theoretic picture, let us think of graded rings and modules as many-sorted algebras, $R = (R_g)_{g\in G}$, $M = (M_g)_{g\in G}$, each "sort" of elements being a homogeneous component. Thus one has multiplication operations $R_g \times R_h \to R_{gh}$, $R_g \times M_h \to M_{gh}$, but no addition of elements of different degrees. Under this definition there are no "nonhomogeneous elements" in M or R; rather these appear when one applies the functors Sum, which take G-graded rings R to ungraded rings $\mathrm{Sum}(R) = \oplus R_g$, and R-modules M to $\mathrm{Sum}(R)$-modules $\mathrm{Sum}(M) = \oplus M_g$. Then we find that $M \mapsto M^G \cong (M^g)_{g\in G}$ is a functor from $\mathrm{Sum}(R)$-modules to R-modules, which has the left adjoint Sum and a right adjoint Prod: $M \mapsto \prod M_g$. For G finite, Sum and Prod coincide. In this paper I have, with misgivings, foregone the above "manysorted" approach and followed the standard the viewpoint which identifies R and $\mathrm{Sum}(R)$, or rather, regards R as $\mathrm{Sum}(R)$ plus some additional structure. Otherwise more explanation would have been needed than the length of the paper would justify. Note that the constructions of going from a G-grading to a G/N-grading etc. can also be described by "Sum"-like functors.

can get a reasonably good hold on these structure questions when G is finite cyclic, allowing us to prove:

Proposition 14. Let G be a finite solvable group, and R a G-graded ring. Then $J_G(R) \subseteq J(R)$.

Proof. Using Lemma 13 we may clearly reduce to the case $G = \mathbb{Z}/p\mathbb{Z}$. Using the idea of Theorem 1, we may assume that R is a $\mathbb{Z}[\omega]$-algebra, where ω is a primitive p^{th} root of unity.

Now let r be a homogeneous element of $R - J(R)$, take a simple (ungraded) R-module M not annihilated by r, and form the graded R-module M^G. Our desired result will follow if we can show that we can form a simple graded factor-module M^G/N, and that this is not annihilated by r. To do this we shall show that as an ungraded R-module M^G has a composition series of length p, each simple factor in which is a copy of M, possibly "twisted", but still not killed by r; then M^G/N will have a simple factor-module M^G/N, and this cannot be annihilated by r. We must consider separately two cases:

Case 1. $pM \neq 0$. Then multiplication by p is invertible on M. Now for each $\mathbb{Z}[\omega]$-valued character χ of $G = \mathbb{Z}/p\mathbb{Z}$, let $M^{\chi} \subseteq M^G$ denote the image of M under the map $x \mapsto (\chi(g)x^g)_{g\in G}$. Unlike the M^g's, the M^{χ}'s will not be homogeneous, but they will be (ungraded) R-submodules. Each is in fact isomorphic to M with its module-structure twisted by one of the automorphisms of R used in the proof of Theorem 1. It is easy to show (using the invertibility of p on M!) that $M^G = \oplus M^{\chi}$, hence M^G is semisimple as an R-module, with the desired sort of composition factors.

Case 2. $pM = 0$. Then define maps $f_i: M \to M^G$ $(0 \leq i < p)$ by $f_i(x) = (\binom{0}{i}x, \ldots, \binom{r}{i}x, \ldots, \binom{p-1}{i}x)$, and let $M^{(i)} = f_0(M)+\ldots+f_i(M)$. The f_i are not module homomorphisms for $i > 0$; but one finds $f_i(ax) \equiv a\, f_i(x) \pmod{M^{(i-1)}}$ $(a\in R)$.

(E.g., if $a \in R_{-1}$, $x \in M$, one finds $f_i(ax) = a(f_i(x) + f_{i-1}(x))$. The general formula is more complicated.) It follows that each $M^{(i)}$ is an R-module, with $M^{(i)}/M^{(i-1)} \cong M$. One also has $M^{(p-1)} = M^G$, so again we have a composition series of the desired sort.‖

Yes. Cohen & Montgomery TAMS 282(1984) 237-258 and 300(1987) 810-811

__4. The problem with nonsolvable G.__ Can the above ~~proof~~ result be extended to more general G? Consider the following reinterpretation of what we have done above. Let R be a G-graded ring. For simplicity, assume R is an algebra over a field K. Form $R \otimes KG$, a $G \times G$-graded ring. Then for every ungraded R-module M, one gets a $\{1\} \times G$-graded $R \otimes KG$-module $M \otimes KG$, and one can study the structure of this module in terms of the structures of M and KG fairly successfully. But this is not quite what we want to do. Let $\tilde{G} \subseteq G \times G$ denote the diagonal subgroup, $\{(g,g) \mid g \in G\}$, and take $\tilde{R} = (R \otimes KG)_{\tilde{G}}$, which will be the subalgebra spanned by all elements $r \otimes g$ where $r \in R_g$. Then $\tilde{R}$ is isomorphic to R, and our M^G is essentially $M \otimes KG$ with scalars restricted to $\tilde{R}$. Now if G is finite and we take a composition series for KG as a left module over itself, $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n$, we get a chain of submodules $M \otimes I_0 \subseteq \ldots \subseteq M \otimes I_n$ of $M \otimes KG$, with factor-modules isomorphic to $M \otimes (I_i/I_{i-1})$, and we may look at these submodules and factor-modules as $\tilde{R}$-modules. When I_i/I_{i-1} is one-dimensional, as is always the case when G is abelian and K has enough roots of 1, then it is easy to describe the $\tilde{R}$-module structure of $M \otimes (I_i/I_{i-1})$, and to see that this is simple if M is so. But when G is noncommutative, there will be terms of dimension > 1, and I don't know what to do with these; certainly $M \otimes (I_i/I_{i-1})$ will not always be simple over $\tilde{R}$.

__5. G-set-graded modules.__ If G is a group and S a left G-set, then one easily sees how to define an S-__graded left R-module__. $M = (M_s)_{s \in S}$. We shall confine our attention to the case $S = G/H = \{gH \mid g \in G\}$, H a subgroup of G. Note that as a left R-module, R can be considered G/H graded for any $H \subseteq G$, and so one can speak of G/H-homogeneous left ideals: left ideals I such that $I = \oplus (I \cap R_{gH})$. We shall write $N(H)$ for the normalizer of H, that is, $\{g \in G \mid g^{-1}H\,g = H\}$. Then it is not hard to prove the equivalence of the descriptions in the following definition. To see that the left ideal described by (1) is closed under right multiplication by $R_{N(G)}$, note that if $r \in R_g$, $g \in N(G)$, and $x \in M_H$ (M a G/H-graded R-module), then $rx \in M_{gH}$, but M can be given a new G/H-grading by putting $N_{aH} = M_{agH}$, and rx is in the basepoint-component of this module.

<u>Definition 14.</u> Let G be a group, H a subgroup, and R a G-graded ring. Then by $J_{G/H}(R)$ we shall mean the $(R, R_{N(H)})$-subbimodule of R characterized equivalently as follows:

(1) The set of elements of R annihilating the "basepoint-component" M_H of every simple G/H-graded R-module M.

(1') The intersection of all maximal G/H-homogeneous left ideals $I \subseteq R$.

(2) The largest G/H-homogeneous left ideal $J \subseteq R$ such that $1+x$ is a unit for all $x \in J \cap R_H$.

(3) The largest G/H-homogeneous left ideal $J \subseteq R$ such that $J \cap R_H \subseteq J(R_H)$.

I suspect that these "radicals" might be useful in an attack on the question of whether $J_G(R) \subseteq J(R)$ for general finite G. Note that for $g \in G$, not necessarily in $N(H)$, one has $J_{G/H}(R)\, R_g \subseteq J_{G/(gHg^{-1})}(R)$. The following concept might also be of use:

<u>Definition 15.</u> For G, H, R as above, we shall denote by $J_{G//H}(R)$ the ideal of R characterized equivalently as follows:

(1) The set of elements of R annihilating every simple G/H-graded R-module M.

(2) $\bigcap_{g \in G} J_{G/gHg^{-1}}(R)$.

REFERENCE: |1| N. Jacobson, <u>Structure of rings.</u>

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