notes on

FUNCTORS FROM FINITE SETS TO FINITE SETS

by George M. Bergman[1]

Introduction

In a course I was teaching this Fall, to help give the students a feel for what functors are like, I proposed the following problem: There are a large variety of covariant and contravariant functors one can construct from the category of finite sets into itself. Any such functor $\underline{F}$ determines a function f from the nonnegative integers to the nonnegative integers, by putting $f(|X|) = |\underline{F}(X)|$ for all finite sets X. What can be said about functions f that arise in this way from functors? Can we characterize them completely?

Working on this problem myself, I eventually obtained a complete characterization of functions arising from contravariant functors and an almost complete characterization in the covariant case; and in the process, some interesting results on the structure of such functors. I have written this up here in rough form. I don't know whether I will rewrite it as an article (with the material better arranged, fuller justification of steps, etc.). I would be interested in any comments on relations of this material to other work,[2] etc..

It will be most convenient to restrict ourselves to the category of nonempty finite sets (with set-maps for morphisms), which we will call FS. We shall follow the set-theoretic convention that a positive integer n is a certain set of cardinality n. If $\underline{F}$: FS→ FS is any covariant or contravariant functor,

[1]The author was partly supported by an Alfred P. Sloan research fellowship.

[2] See addendum, p. 12.

then for each integer n, consider the set of elements $u \in \underline{F}(n)$ which are not images under any maps $\underline{F}(a)$ of elements $v \in \underline{F}(n-1)$ ($a \in \mathrm{Hom}(n-1, n)$ in the covariant case, $a \in \mathrm{Hom}(n, n-1)$ in the contravariant case). Now $\underline{F}$ induces an action of the symmetric group $S_n = \mathrm{Aut}_{\underline{FS}}(n)$ on $\underline{F}(n)$. Hence to such a u, we can associate its isotropy subgroup $G \subseteq S_n$. We shall find that G almost completely determines the behavior of u under maps $\underline{F}(b)$ ($b \in \mathrm{Hom}(n,X)$ in the covariant case, $b \in \mathrm{Hom}(X,n)$ in the contravariant case; $X \in \mathrm{Ob}(\underline{FS})$.)

Below, we shall begin by obtaining for every n and every subgroup $G \subseteq S_n$ a functor $P_{n,G}$: $\underline{FS} \to \underline{FS}$ (in the contravariant case; $Q_{n,G}$ in the covariant case) as suggested by the above observations, and show that these functors are "building blocks" from which all endofunctors of $\underline{FS}$ can be constructed by an extension process. The numerical functions associated with these building blocks come from binomial coefficients in the covariant case, and Stirling numbers in the contravariant case. In the latter case a straightforward construction will show that the $P_{n,G}$'s can occur in a functor $\underline{F}$ with arbitrary multiplicities. The multiplicities do not generally completely determine $\underline{F}$, but they do determine the associated numerical function, solving our initial problem. In the contravariant case, conditions are slightly more complicated and our results just miss constituting a precise criterion.

We end with a few observations on the structures of such functors $\underline{F}$. The have a strong similarity to simplicial complexes, with the "building-block" functors in the role of simplices.

Contravariant functors

Given any $A \in \underline{FS}$, and any subgroup $G \subseteq S_A =_{def} \mathrm{Aut}_{\underline{FS}}(A)$, let us define a contravariant functor $P'_{A,G}: \underline{FS} \to \underline{FS}$ by putting $P'_{A,G}(X) = \mathrm{Hom}(X, A)/E(G)$, where $E(G)$ is the equivalence relation on $\mathrm{Hom}(X, A)$ which has one equivalence class consisting of all the nonsurjective maps $X \to A$, and such that the equivalence classes of surjective maps are the orbits under the induced action of G. When $|A| = 1$, the first-mentioned "equivalence class" is actually empty, but we shall throw in an extra "basepoint" anyway. It is easy to see how to define $P'_{A,G}(a): P'_{A,G}(Y) \to P'_{A,G}(X)$ for any map $a: X \to Y$ in $\underline{FS}$.

Let $|A| = k$. Then the integer-valued function associated to $P'_{A,G}$ can be written $1 + (S_A:G)\, p_k(n)$, where

$$p_k(n) = \frac{1}{k!}\left(k^n - \binom{k}{1}(k-1)^n + \ldots \pm \binom{k}{k-1}\right)$$

= the number of partitions of a set of cardinality n into exactly k nonempty subsets. (These are Stirling numbers of the second kind. Without the $1/k!$, the formula represents the number of surjective maps of n to k; it is obtained by induction from the formula k^n for the total number of maps from n to k.)

Note that the functors $P'_{A,G}$ can be thought of as functors from $\underline{FS}$ into the category of finite sets with basepoint. By taking sums-with-amalgamation-of-basepoint we see that we can get a functor giving any numerical function of the form $1 + c_1\, p_1(n) + c_2\, p_2(n) + \ldots$, where the c_i are nonnegative integers, (which we do not require to be almost all zero, because for each n, only $p_1,\dots,p_n$ have nonzero value at n, so the functors we obtain by such infinite sums are still FS-valued. Incidentally, note that $p_1(n)=1$.)

We shall show that conversely, every contravariant functor $\underline{F}: \underline{FS} \to \underline{FS}$ induces a function f of this form. $\underline{F}$ need not be an infinite "sum" of functors $P_{A,G}$ as above, but we shall see that it is in general the limit of a sequence of extensions of functors $P_{A,G}$.

It is clear how to define a subfunctor of an FS-valued functor, and if $\underline{F}'$ is a subfunctor of $\underline{F}$, we define the quotient functor $\underline{F}/\underline{F}'$ to associate to each X the set $\underline{F}(X)$ with the subset $\underline{F}'(X)$ identified to a point. We shall say that the functor $\underline{F}$ is an extension of $\underline{F}'$ by $\underline{F}/\underline{F}'$. Note that if in this situation, the functions associated with F, F', $\underline{F}/\underline{F}'$ are written 1+u, 1+v, 1+w, then u = v+w. The "sums with amalgamation" considered above are a special case of extensions of functors. (As another example, the functor $X \to k^X$ is an extension by $P_{k,\{e\}}$ of the functor associating to X the set of all $a \in \mathrm{Hom}(X,k)$ with $|a(X)| \leq n-1$, which in turn is an extension by the "sum" of $\binom{k}{1}$ copies of $P_{k-1,\{e\}}$ of the functor giving the set of a such that $|a(X)| \leq n-2$, etc.).

Let us now fix a contravariant functor $\underline{F}: \underline{FS} \to \underline{FS}$. Given any nonempty set $S \subseteq \coprod_X \underline{F}(X)$ (X ranging over some family of sets), we can define the subfunctor of $\underline{F}$ generated by S as the functor sending each Y to the set of all elements of $\underline{F}(Y)$ which are images of elements of S under the various maps $\underline{F}(a)$ ($a \in \mathrm{Hom}(Y,X)$, $X \in \underline{FS}$). This will be nonempty for all Y because Hom(Y,X) is always nonempty!

For k = 1, 2, ..., let us define $\underline{F}^{(k)}$ to be the subfunctor of $\underline{F}$ generated by all of $\underline{F}(k)$.* Since the identity map of k-1 factors through k, we get $\underline{F}^{(1)} \subseteq \underline{F}^{(2)} \subseteq \dots$, with $\underline{F}$ as the union.

Lemma $\underline{F}^{(1)}$ is a sum (as functors-with-basepoints) of a finite family of copies of $P_{1,\{e\}}$. $\underline{F}^{(k)}/\underline{F}^{(k-1)}$ $(k>1)$ is likewise the sum of a finite family of functors of the form $P_{k,G}$, for various subgroups G of S_k.

Proof. Since 1 is the final object of $\underline{FS}$, hence the initial object of $\underline{FS}^{o}$, $\underline{F}^{(1)}(X)$ will consist of a canonical image in $\underline{F}(X)$ of $\underline{F}(1)$, for each X. Using the fact that there are also maps of every element of $\underline{FS}^{o}$ back to 1, it is easily deduce that $\underline{F}^{(1)}$ is isomorphic to the constant functor with value $\underline{F}(1)$. Say $\underline{F}(1)$ has n elements; then $\underline{F}^{(1)}$ can be represented as the "sum" with amalgamation of basepoints of n-1 copies of $P_{1,\{e\}}$.

Now let $k > 1$, $\underline{H} = \underline{F}^{(k)}/\underline{F}^{(k-1)}$. We note that the collapsed subfunctor $\underline{F}^{(k-1)} \subseteq \underline{F}^{(k)}$ induces basepoints $o_X \in \underline{H}(X)$, that $\underline{H}(j) = \{o_j\}$ for $j < k$, and that $\underline{H}$ is generated by $\underline{H}(k)$. We shall show that any functor $\underline{H}$ with these properties is a "sum" of functors ~~[illegible]~~ of the form $P_{j,H}$.

Let $U \subseteq \underline{G}(k)$ be a set of representatives for the orbits other than $\{o_k\}$ under the action of $\underline{G}(S_k)$ on $\underline{G}(k)$. Denote the isotropy subgroup of each $u \in U$ by $G_u \subseteq S_k$. To prove our claim it will suffice to show:

(1) Given $u \in U$, $X \in \underline{FS}$, $a \in Hom(X, k)$, we have $\underline{G}(a)(u) = o_X \iff a$ is not surjective.

(2) Given $u, u' \in U$, $X \in \underline{FS}$, and surjective $a, a' \in Hom(X,k)$, if $\underline{G}(a)(u) = \underline{G}(a')(u')$, then $u = u'$.

(3) Given $u \in U$, $X \in \underline{FS}$, and surjective $a, a' \in Hom(X,k)$, we have $\underline{G}(a)(u) = G(a')(u) \iff a' \in H_u a$.

(1) and (2) are simple. In (1), to get "$\Leftarrow$" note that a nonsurjective map a can be factored through k-1, and $\underline{G}(k-1) = \{o_{k-1}\}$. To get "$\Rightarrow$" note that if a is surjective, it is right invertible. In (2), we choose a right inverse b to a. Applying $\underline{G}(b)$ to the equation $\underline{G}(a)(u) = \underline{G}(a')(u')$, we get $\underline{G}(ab)(u) = u'$. By (1), this means $ab \in S_k$, and since U is a set of orbit representatives, we must have u=u'.

In (3), "$\Leftarrow$" is trivial, but "$\Rightarrow$" is a bit surprising. The idea of the is: in order for the element u to be able to distinguish between surjective and nonsurjective maps of sets into k (which we know it does by (1)), it must also be able to distinguish among surjective maps that differ by more than the action of the automorphisms of k. Indeed, suppose $\underline{G}(a)(u) = \underline{G}(a')(u)$. As in the proof of (2), we choose $b: k \to X$ with $a'b = I_k$, and find $\underline{G}(ab)(u) = u$, so $ab \in H_u \subseteq S_k$. It will clearly suffice to prove the elements a' and $a'' = (ab)^{-1}a \in H_u a$ equal.

Note that a' and a'' both have b as a right inverse, and satisfy $\underline{G}(a')(u) = \underline{G}(a'')(u)$. Suppose $a' \neq a''$; choose $x \in X$ with $a'(x) \neq a''(x)$. Define $c \in \mathrm{Hom}(k,X)$ to agree with b except at $a'(x)$, and take this to x. Thus $a'c$ and $a''c$ both equal the identity function except, perhaps, at $a'(x)$. The former, in fact, leaves this element fixed as well, so $a'c = I_k$, while the latter takes $a'(x)$ to $a''(x)$. Since it differs from the identity at exactly one value, it must be nonsurjective. Hence $o_k = \underline{G}(a''c)(u) = \underline{G}(c)\underline{G}(a'')(u) = \underline{G}(c)\underline{G}(a')(u) \neq o_k$ — contradiction!

This completes the proof of the Lemma; and our assertions about obtaining arbitrary functors as extensions of the $P_{k,G}$ follow. We hence get:

<u>Corollary</u> A function on the positive integers is induced by some contravariant functor $\underline{FS} \to \underline{FS}$ if and only if it is of the form $1 + c_1p_1 + c_2p_2 + \ldots$ ($c_k \geq 0$, p_k as on p.3.) In particular, for f such an induced function, (a) if $f \leq c\, k^n$ for some c, k, then f is a linear combination with rational coefficients of the functions $1, 2^n, \ldots, k^n$, while (b) if $f(1)=f(2)=\ldots=f(k-1)\neq f(k)$, then $f(n) > p_k(n) \approx k^n/k!$

<u>Comment</u>: Let us turn back to the point where we defined the functors $P_{A,G}$, and suppose we had not decided to "adjoin a basepoint" in the case $|A| = 1$. We

would have gotten functors $P'_{A,G}$, agreeing with $P_{A,G}$ for $|A| > 1$, but with $P'_{A,G} \cong P'_{1,\{e\}}$ being the trivial (one-element constant) functor for $|A| = 1$. In the above development, we took cunning advantage of the fact that the the functor $P'_{1,\{e\}}$ equalled the trivial functor: On the one hand, it absolved us from considering separately the functor $P'_{1,\{e\}}$ which does not fit nicely into our family of $P_{A,G}$'s, because we started our constructions with the "basepoint" functor; on the other hand, we didn't have to worry about disjoint sums of more than one copy of the trivial functor, because we get these as sums-with-amalgamation-of-basepoint of copies of $P_{1,\{e\}}$.

In the covariant case, we shall have a similar family of building-block functors $Q_{A,G}$, and again, all have a basepoint except for the functor we get when $|A|=1$. But in this case, $Q_{1,\{e\}}$ will not be the trivial functor but the identity functor. We shall find that there exist two "minimal" functors under the partial ordering "is embeddable in": the trivial functor, which can be written $Q_{0,\{e\}}$, and the identity functor, $Q_{1,\{e\}}$. This fact will prevent us from obtaining a complete solution to our problem, in the

<u>Covariant case</u> Given any finite set A, including the empty set, and any subgroup G of Aut(A), we get a covariant functor $Q_{A,G}$: <u>FS</u>$\rightarrow$ <u>FS</u> by putting $Q_{A,G}(X) = \mathrm{Hom}(A,X)/E$, where E identifies orbits under the action of G, and lumps all noninjective maps together. In this case we will not adjoin a basepoint in the cases $|A|=0,1$ where all maps are injective; and we find the associated integer-valued functions are:

$$i_G \binom{n}{k} \text{ for } k=0,1$$

$$1+ i_G \binom{n}{k} \text{ for } k \geq 2.$$

We will have to work both with functors with and without basepoints. In particular, we see how to define an extension of an arbitrary functor by a functor with basepoint, to get an arbitrary functor. E.g., the functor Hom(2,-) is an extension of $Q_{1,\{e\}}$ (without basepoint) by $Q_{2,\{e\}}$

(with basepoint). Note that by taking "sums with amalgamation of basepoint" of functors $Q_{A,G}$, $|A| \geq 2$, and throwing in disjointly copies of $Q_{0,\{e\}}$, $Q_{1,\{e\}}$, we can realize every function of the form $c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \ldots$ with $c_0 > 0$, all other $c_i \geq 0$. But there are other functions we can obtain, e.g., $\binom{n}{1}$.

Given a functor $\underline{F}$: $\underline{FS} \rightarrow \underline{FS}$, we define $\underline{F}^{(i)}$ $(i=1,2,\ldots)$ as in the covariant case. The analog of our earlier lemma can be broken into two parts, one of which differs from the contravariant case, and one of which is quite analogous to what is true in that case. The different part is:

Lemma $\underline{F}$ is the union of a finite family of disjoint subfunctors, each of which contains exactly one element of $\underline{F}(1)$. Further, a functor $\underline{G}$ generated by a single element $u \in \underline{G}(1)$ is isomorphic to either $Q_{0,\{e\}}$ or $Q_{1,\{e\}}$, hence each of the subfunctors into which we have partitioned $\underline{F}$ has a unique minimal subfunctor of one of these forms.

Proof First assertion: 1 is the final element of $\underline{FS}$, so every element of any $\underline{F}(X)$ has a unique image in 1. By partitioning the elements of these sets according to their images in 1, we get the desired partition of the functor $\underline{F}$. Second assertion: it is easy to show that if the two maps of 1 into 2 send u to the same element of $\underline{G}(2)$, then $\underline{G} \cong Q_{0,\{e\}}$, while if they send it to two different elements, $\underline{G} \cong Q_{1,\{e\}}$.

Lemma For $k > 1$, $\underline{F}^{(k)}/\underline{F}^{(k+1)}$ is a sum as functors-with-basepoint of a finite family of copies of $Q_{k,G}$, for various subgroups G of S_k.

Proof Exactly analogous to the contravariant case!

Corollary For a function on the positive integers to be induced by a covariant functor $\underline{FS} \rightarrow \underline{FS}$, a necessary condition is that it be of the form $c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \ldots$ with all $c_k \geq 0$, and c_0 or c_1 positive. A sufficient condition is that it have this form with c_0 positive. In particular, for f such an induced function, (a) if $f \leq c\, n^k$ for some k, then f is a polynomial function of degree $\leq k$, and (b) if $f(1)=f(2)=\ldots=f(k-1)\neq f(k)$ $(k>1)$, then $f(n) \geq n$ if $k=2$, $f(n) \geq \binom{n}{k}$ if $k>2$.

In the contravariant case, our functions p_k were linearly independent, each taking its first nonzero value at n=k. In this case, $\binom{n}{0}$ and $\binom{n}{1}$ both take their first nonzero value (for positive n) at n=1, and in fact, there is precisely one "dependence relation" (not a linear dependence relation in the usual sense because it involves infinite sums):

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots\ .$$

And in fact, each side of this relation can be represented by a functor: given any $X \in \underline{FS}$, let $\underline{B}(X)$ denote the set of all $\mathbb{Z}/2\mathbb{Z}$-valued functions on X, and given $f: X \to Y$, and $u \in \underline{B}(X)$, define $\underline{B}(f)(u) \in \underline{B}(Y)$ by $\underline{B}(f)(u)(y) = \sum_{x \in f^{-1}(y)} u(x)$.

If we think of $\underline{B}(X)$ as consisting of all subsets of X, then we see that for $u \in \underline{B}(X)$, the parity of $|\underline{B}(f)(u)|$ is the same as that of $|u|$. It follows that $\underline{B}$ decomposes as the disjoint union of two functors $\underline{B}_{odd}$ and $\underline{B}_{even}$, which, respectively, associate to X the set of all its subsets of odd cardinality, and the set of its subsets of even cardinality. Each of these induces the same integer-valued function, 2^{n-1}, but it is easy to see that the first is the union of a chain of extensions whose successive factors have functions $\binom{n}{1}$, $\binom{n}{3}$, etc., while the second gives $\binom{n}{0}$, $\binom{n}{2}$, $\binom{n}{4}$, ...

(It might seem that this nonuniqueness could be cured by considering functors on the category of all finite sets, i.e., bringing in $\emptyset$. While it is true that $\underline{B}_{odd}$ and $\underline{B}_{even}$ would then differ on $\emptyset$, there exists a functor

$\underline{E}$ such that $|\underline{E}(\emptyset)| = 2$, $|\underline{E}(A)| = 1$ for $|A| > 0$, and we see that $|\underline{E} + \underline{B}_{odd}|$ induces the same function as $|\underline{Q}_{0,\{e\}} + \underline{B}_{even}|$. While we're on the subject, note that the difficulty in getting an exact criterion for an integer-valued function to be covariantly induced would disappear if we considered functors into the category of finite sets with basepoint. The answer would then be precisely the condition given as "sufficient" in the last Lemma. It appears that we would get equally complete results (slightly different in detail) if we made the domain category finite pointed sets, and the range either $\underline{FS}$ or the same as the domain. Note also that the structure-theoretic (as distinct from the numerical) parts of the above results are equally valid without the finiteness condition on the sets of the range category.)

For any odd positive integer $2k+1$, we can get the function $\binom{n}{1}+\binom{n}{3}+\ldots+\binom{n}{2k+1}$, from $\underline{B}_{odd}^{(2k+1)}$.

For any positive integer k, we get the function $\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{k}$ from $\underline{C}^{(k)}$, where $\underline{C}$ associates to X the set of all nonempty subsets of X, and to a map $f\colon X \to Y$ the map taking $u \in \underline{C}(X)$ to $f(u) \in \underline{C}(Y)$.

For any $k > 1$, let $S_{k-1} \subseteq S_k$ denote the isotropy subgroup of the element 0. Form the quotient of $\mathrm{Hom}(k, X)$ by the action of S_{k-1}, and then identify any two noninjective maps if they agree on $0 \in k$. The resulting functor is an extension of $Q_{1,\{e\}}$ by $\underline{Q}_{k,S_{k-1}}$, and it induces the function $\binom{n}{1} + k\binom{n}{k}$.

All this serves to remind us that the really interesting problem is not that of integer-valued functions, but the study of functors from $\underline{FS}$ to $\underline{FS}$. The integer-valued functions gave us a sufficiently specific and tractable "test-question" to lead us to discover some basic structure-theory of these functors. We can now see that, in both the covariant and contravariant cases, a basic problem is how to describe the extensions that can exist among the functors $P_{A,G}$ and $Q_{A,G}$. We can make a few observations on this point.

Given A and G as usual, let us define $\tilde{\underline{P}}_{A,G}$ and $\tilde{\underline{Q}}_{A,G}$ as the functors obtained by dividing $\mathrm{Hom}(-, A)$ or $\mathrm{Hom}(A, -)$ by the action of G (but not identifying all nonsurjective or noninjective maps. If we had done things right we would have started with these functors and obtained $\underline{P}_{A,G}$ and $\underline{Q}_{A,G}$ from them.) It is easy to check:

Lemma Let A and B be disjoint sets, and G, H subgroups of S_A and S_B. Then

$$\tilde{\underline{P}}_{A\times B,\ G\times H} \cong \tilde{\underline{P}}_{A,G} \times \tilde{\underline{P}}_{B,H}, \qquad \text{and} \qquad \tilde{\underline{Q}}_{A\cup B,\ G\times H} \cong \tilde{\underline{Q}}_{A,G} \times \tilde{\underline{Q}}_{B,H}.$$

If $|A| = k > 1$, define $\partial\tilde{\underline{P}}_{A,G} = \tilde{\underline{P}}^{(k-1)}_{A,G}$, and $\partial\tilde{\underline{Q}}_{A,G} = \tilde{\underline{Q}}^{(k-1)}_{A,G}$. Then we see that $\underline{P}_{A,G} = \tilde{\underline{P}}_{A,G}/\partial\tilde{\underline{P}}_{A,G}$ and $\underline{Q}_{A,G} = \tilde{\underline{Q}}_{A,G}/\partial\tilde{\underline{Q}}_{A,G}$. Further, we find:

Lemma Extensions $\underline{E}$ of a functor $\underline{F}$ by $\underline{P}_{A,G}$ (resp. $\underline{Q}_{A,G}$) are in natural 1-1 correspondence with morphisms of functors g: $\tilde{\underline{P}}_{A,G}$ (resp. $\tilde{\underline{Q}}_{A,G}$) $\to \underline{F}$. To be precise, any such $\underline{E}$ can be obtained in a unique manner as the pushout of a diagram:

$$\begin{array}{ccc} \partial\tilde{\underline{P}}_{A,G} & \xrightarrow{g} & \underline{F} \\ \downarrow & & \vdots \\ \tilde{\underline{P}}_{A,G} & \dashrightarrow & \underline{E} \end{array} \qquad \left(\text{resp.} \quad \begin{array}{ccc} \partial\tilde{\underline{Q}}_{A,G} & \xrightarrow{g} & \underline{F} \\ \downarrow & & \vdots \\ \tilde{\underline{Q}}_{A,G} & \dashrightarrow & \underline{E} \end{array}\right).$$

The analogy with geometry is striking: The $\tilde{\underline{P}}$'s and $\tilde{\underline{Q}}$'s are "simplices" or "discs", the $\underline{P}$'s and $\underline{Q}$'s "spheres", $\underline{F}^{(k)}$ is the "k-skeleton" of $\underline{F}$. But we have different kinds of "k-simplices" corresponding to different subgroups $G \subseteq S_k$!

This relationship with geometry can, in part, be made precise. Let $\underline{SC}$ denote the category of finite simplicial complexes and simplicial maps. Then the full subcategory of all simplices is equivalent to the category of finite sets, since a map between simplices is just a set-map of the vertices. Hence from every $C \in \mathrm{Ob}(\underline{SC})$ we get a contravariant functor from $\underline{\text{Simplices}} \cong \underline{FS}$ into $\underline{FS}$, defined by $\mathrm{Hom}(-, C)$. The functors $\underline{FS} \to \underline{FS}$ so realizable are precisely those whose "components" $P_{k,G}$ under our decomposition all have $G = \{e\}$ and are connected by boundary-maps (as in the preceding Lemma) which are all 1-1. In fact, the full subcategory of functors of this sort in Funct($\underline{FS}^{op}$, $\underline{FS}$) is equivalent to $\underline{SC}$. (If one is willing to stretch things, one can even enlarge $\underline{SC}$ to a category of "geometric" objects which represent arbitrary members of Funct($\underline{FS}^{op}$, $\underline{FS}$). On the other hand, if we take a simplicial complex C and look at the covariant functor Hom(C, –) on simplices, we find that for any simplex Δ, $\mathrm{Hom}(C, \Delta) \cong \mathrm{Hom}(C^{(0)}, \Delta^{(0)})$, so we neither obtain interesting information about C, nor a large class of covariant functors. We have not been able to find any geometric realization of general covariant functors.)

It would be interesting to see whether homotopy and other geometric concepts have analogs for these functors.

Winter 1971-72

Addendum. March, 1973.

Here are some related references that have been brought to my attention. Items |1|-|4| study covariant and contravariant endofunctors of the category S of all sets. The question studied in |5| is analogous to that studied here. In §VII.5 of |6|, "simplicial objects" in an arbitrary category C are developed as contravariant C-valued functors on (a skeleton of) the category of finite totally ordered sets.

|1| V. Trnková, Some properties of set functors, Comment.Math.Univ.Carolinae 10(1969), 323-352.

|2| V. Trnková, On descriptive classification of set functors, Comment.Math. Univ.Carolinae, part I: 12(1971)143-175, part II: to appear.

|3| V. Koubek, Set functors, Comment Math.Univ.Carolinae 12(1971)175-195.

|4| V. Koubek, (to appear, same journal).

|5| D.B.A. Epstein and M Kneser, Functors between categories of vector spaces, pp.154-170 Category Theory, Homology and their applications, III, Springer Lecture Notes 99.

|6| S. MacLane, Categories for the Working Mathematician, Springer Graduate Texts in Mathematics, 5, 1971.

Addendum. May, 2010.

The covariant case, which I was able to solve here only up to a possible added term +1, is completely solved in:

|7| R. Daugherty, Functors on the category of finite sets, TAMS 330 (1992) 859-886. MR 92f:18001.

The MR review of that paper points to further related work.