ON DIAGRAM-CHASING IN DOUBLE COMPLEXES

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Abstract. We construct, for any double complex in an abelian category, certain "short-distance" maps, and an exact sequence involving these, instances of which can be pieced together to give the "long-distance" maps and exact sequences of results such as the Snake Lemma.

Further applications are given. We also note what the building blocks of an analogous study of triple complexes would be.

Introduction

Diagram-chasing arguments frequently lead to "magical" relations between distant points of diagrams: exactness implications, connecting morphisms, etc.. These long connections are usually composites of short "unmagical" connections, but the latter, and the objects they join, are not visible in the proofs. This note is aimed at remedying that situation.

Given a double complex in an abelian category, we will consider, for each object A of the complex, the familiar horizontal and vertical homology objects at A (which we will denote $A{\text{-}\!\bullet\!\text{-}}$ and $A{\updownarrow}$), and two other objects, $A_{\square}$ and ${}^{\square}\!A$ (which we name the "donor" and the "receptor" at A). For each arrow of the double complex, we prove in §1 the exactness of a certain 6-term sequence of maps between these objects (the "Salamander Lemma"). Standard results such as the 3×3-Lemma, the Snake Lemma, and the long exact sequence of homology associated with a short exact sequence of complexes are recovered in §§2-4 as easy applications of this result.

In §5 we generalize the last of these examples, getting various exact diagrams from double complexes with all but a few rows and columns exact. The total homology of a double complex is examined in §6.

In §7.2 we take a brief look at the world of triple complexes, and in §7.3 at the relation between the methods of this note and J. Lambek's homological formulation of Goursat's Lemma [8]. We end with a couple of exercises.

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1. Definitions, and the Salamander Lemma

We shall work in an abelian category **A**. In the diagrams we draw, capital letters and boldface dots $\bullet$ will represent arbitrary objects of **A**. (Thus, such a dot does not imply a zero object, but simply an object we do not name.) Lower-case letters and arrows will denote morphisms. When we give examples in categories of modules, "module" can mean left or right module, as the reader prefers.

A *double complex* is an array of objects and maps in **A**, of the form

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & {} \\
& {} & {} & {} & {} &
\end{tikzcd} \tag{1}$$

extending infinitely on all sides, in which every row and every column is a complex (i.e., successive arrows compose to zero), and all squares commute. Note that a "partial" double complex such as

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd} \tag{2}$$

can be made a double complex by completing it with zeroes on all sides; or by writing in some kernels and cokernels, and then zeroes beyond these. Thus, results on double complexes will be applicable to such finite diagrams.

Topologists often prefer double complexes with *anti*commuting squares; but either sort of double complex can be turned into the other by reversing the signs of the arrows in every other row. In the theory of spectral sequences, vertical arrows generally go upward, while in results like the Four Lemma they are generally drawn downward; I shall follow the latter convention.

1.1. Definition. *Let A be an object of a double complex, with nearby maps labeled as shown below:*

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "b"'] & \bullet \arrow[r] \arrow[d, "c"'] & \bullet \arrow[d] \\
\bullet \arrow[r, "d"] \arrow[d] & A \arrow[r, "e"] \arrow[d, "f"'] & \bullet \arrow[d, "g"'] \\
\bullet \arrow[r] & \bullet \arrow[r, "h"] & \bullet
\end{tikzcd}, \quad \textit{and let } p = ca = db, \ \ q = ge = hf. \tag{3}$$

Then we define

$$
\begin{aligned}
A_{\to} &= \text{Ker } e \,/\, \text{Im } d, \ \textit{the horizontal homology object at } A,\\
A_{\downarrow} &= \text{Ker } f \,/\, \text{Im } c, \ \textit{the vertical homology object at } A,\\
{}^{\square}A &= (\text{Ker } e \cap \text{Ker } f) \,/\, \text{Im } p, \ \textit{which we shall call the } \text{receptor } \textit{at } A,\\
A_{\square} &= \text{Ker } q \,/\, (\text{Im } c + \text{Im } d), \ \textit{which we shall call the } \text{donor } \textit{at } A.
\end{aligned}
$$

From the inclusion relations among the kernels and images in Definition 1.1, we get

1.2. Lemma. *For every object A of a double complex, the identity map of A induces a commuting diagram of maps:*

$$
\begin{tikzcd}
& {}^{\square}A \arrow[dl] \arrow[dr] & \\
A_{\downarrow} \arrow[dr] & & A_{\to} \arrow[dl] \\
& A_{\square} &
\end{tikzcd}
\qquad \blacksquare \tag{4}
$$

1.3. Definition. *We shall call the maps shown in* (4) *the* intramural *maps associated with the object A.*

When we draw the diagram of a double complex, the donor and receptor at an object will generally be indicated by small squares to the lower right and upper left of the dot or letter representing that object, as in (5) below. Thus, the direction in which the square is displaced from the letter is toward the most distant point of the diagram involved in the definition of the object; in (3), the domain or codomain of the composite arrow p or q. (Of course, if one prefers to draw double complexes with arrows going *upward* and to the right, one should write ${}_{\square}A$ and $A^{\square}$ for the receptor and donor at A.)

I will occasionally indicate horizontal or vertical homology objects in a diagram by marks $\to$ and $\downarrow$ placed at the location of the object; but this requires suppressing the symbol for the object itself.

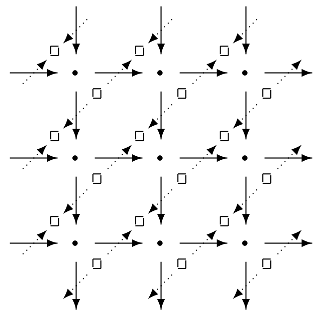
The next result, whose proof is again straightforward, motivates the names "donor" and "receptor".

1.4. Lemma. *Each arrow $f : A \to B$ in a double complex induces an arrow $A_{\square} \to {}^{\square}B$:*

$$
\begin{tikzcd}
& \arrow[d] & \arrow[d] & \\
\arrow[r] & A \arrow[r] \arrow[d] & {}^{\square}B \arrow[r] \arrow[d] & {} \\
& A_{\square} \arrow[ur, dotted] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \arrow[d] & \\
\arrow[r] & A \arrow[r] \arrow[d] & {} \\
& A_{\square} \arrow[d, dotted] & \\
\arrow[r] & {}^{\square}B \arrow[r] \arrow[d] & {} \\
& {} &
\end{tikzcd}
\qquad \blacksquare \tag{5}
$$

1.5. Definition. *We shall call the morphism of Lemma 1.4 the* extramural *map associated with f.*

The global picture of the extramural maps in a double complex is



(6)

In most of this note, I shall not use any notation for these intramural and extramural maps. Between any two of the objects we have constructed, we will not define more than one map, so we shall be able to get by with an unlabeled arrow, representing the unique map constructed between the objects named. (In §6, where we will have more than one map between the same pair of objects, I will introduce symbols for some of these.)

To show a *composite* of the maps we have defined, we may use a long arrow marked with dots indicating the intermediate objects involved, as in the statement of the following easily verified lemma.

1.6. Lemma. *If* $f : A \to B$ *is a* horizontal *arrow of a double complex* (*as in the first diagram of* (5)), *then the natural induced map between* vertical *homology objects,* $A{\downarrow} \to B{\downarrow}$, *is the composite of one extramural, and two intramural maps:*

$$A{\downarrow} \longrightarrow \underset{A_\square}{\bullet} \longrightarrow \overset{{}^\square B}{\bullet} \longrightarrow B{\downarrow}\,. \tag{7}$$

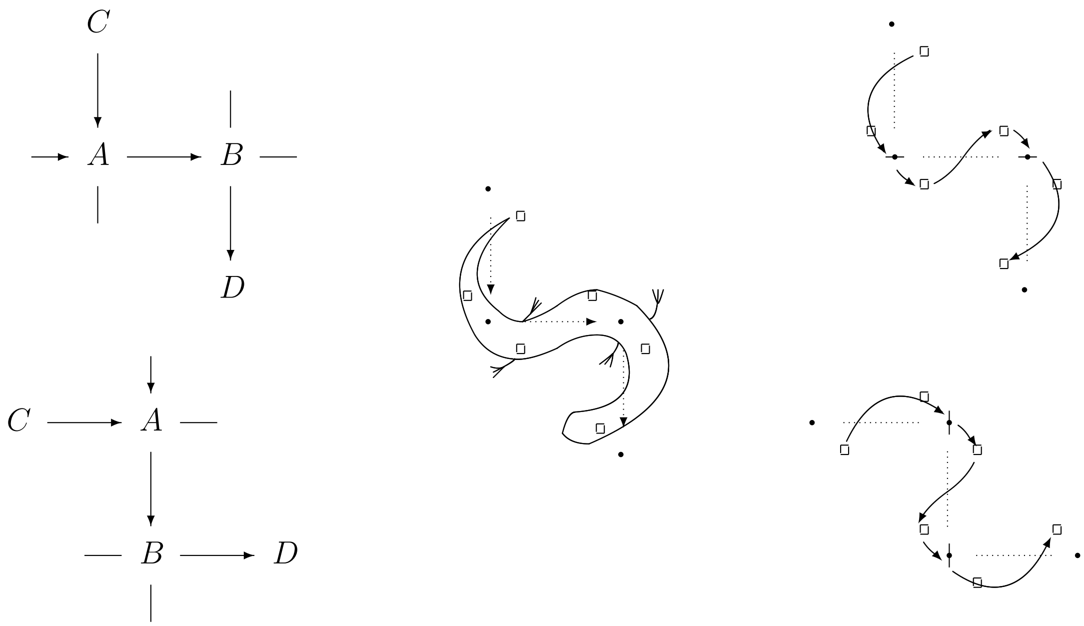
Similarly, for a vertical *map* $f : A \to B$ (*as in the second diagram of* (5)), *the natural induced map of* horizontal *homology objects is given by*

$$A{\leftarrow} \longrightarrow \underset{A_\square}{\bullet} \longrightarrow \overset{{}^\square B}{\bullet} \longrightarrow B{\leftarrow}\,. \tag{8}$$

■

We now come to our modest main result. We will again state both the horizontal and vertical cases, since we will have numerous occasions to use each. The verifications are trivial if one is allowed to "chase elements". To get the result in a general abelian category, one can use one of the concretization theorems referred to at [13, Notes to Chapter VIII], or the method of generalized "members" developed in [13, §VIII.3], or the related method discussed at [2]. (Regarding point (vi) of [13, §VIII.4, Theorem 3], we note that this might be replaced by the following more convenient statement, clear from the proof: *Given* $g : B \to C$, *and* $x, y \in_m B$ *with* $gx \equiv gy$, *there exist* $x' \equiv x$, $y' \equiv y$ *with a common domain, such that* $gx' = gy'$.)

We give our result a name analogous to that of the Snake Lemma.



(9)

1.7. Lemma. [Salamander Lemma.] *Suppose $A \to B$ is a horizontal arrow in a double complex, and C, D are the objects above A and below B respectively, as in the upper left diagram of* (9)*. Then the following sequence* ((9)*, upper right*)*, formed from intramural and extramural maps, is exact:*

$$C_{\square} \xrightarrow{\;{}^{\square}\!A\;} A{\cdot\!\!-} \longrightarrow A_{\square} \longrightarrow {}^{\square}\!B \longrightarrow B{\cdot\!\!-} \xrightarrow[\;B_{\square}\;]{} {}^{\square}\!D. \tag{10}$$

Likewise, if $A \to B$ is a vertical arrow ((9)*, lower left*)*, we have an exact sequence* ((9)*, lower right*)*:*

$$C_{\square} \xrightarrow{\;{}^{\square}\!A\;} A\dot{|} \longrightarrow A_{\square} \longrightarrow {}^{\square}\!B \longrightarrow B\dot{|} \xrightarrow[\;B_{\square}\;]{} {}^{\square}\!D. \tag{11}$$

■

In each case, we shall call the sequence displayed "the 6-term exact sequence associated with the map $A \to B$ of the given double complex".

Remark: For $A \to B$ a horizontal arrow in a double complex, and C, D as in the upper left diagram of (9), not only (10) but also (11) makes sense, but only the former is in general exact. Indeed, by Lemma 1.6, the middle three maps of (11) compose in that case to the natural map from $A\dot{|}$ to $B\dot{|}$, rather than to zero as they would if it were an exact sequence. Likewise, if $A \to B$ is a vertical map, then (10) and (11) both make sense, but only the latter is in general exact.

2. Special cases, and easy applications

Note that the extramural arrows in (6) stand head-to-head and tail-to-tail, and so cannot be composed. This difficulty is removed under appropriate conditions by the following corollary to Lemma 1.7, which one gets on assuming the two homology objects in (10) or (11) to be 0 :

2.1. Corollary. *Let $A \to B$ be a horizontal (respectively, vertical) arrow in a double complex, and suppose the row (resp., column) containing this map is exact at both A and B. Then the induced extramural map $A_{\square} \to {}^{\square}B$ is an isomorphism.* ∎

Using another degenerate case of Lemma 1.7, we get conditions for *intramural* maps to be isomorphisms, allowing us to identify donor and receptor objects with classical homology objects.

2.2. Corollary. *In each of the situations shown below, if the diagram is a double complex, and the darkened row or column (the row or column through B perpendicular to the arrow connecting it with A) is exact at B, then the two intramural maps at A indicated above the diagram are isomorphisms.*

$$
\begin{array}{cccc}
{}^{\square}A \xrightarrow{\cong} A\!- & {}^{\square}A \xrightarrow{\cong} A\!\mid & A\!- \xrightarrow{\cong} A_{\square} & A\!\mid \xrightarrow{\cong} A_{\square} \\
A\!\mid \xrightarrow{\cong} A_{\square} & A\!- \xrightarrow{\cong} A_{\square} & {}^{\square}A \xrightarrow{\cong} A\!\mid & {}^{\square}A \xrightarrow{\cong} A\!-
\end{array}
$$

$$
\begin{tikzcd}[column sep=small]
 & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & \cdot \arrow[r, dash] \arrow[d] & {} \\
0 \arrow[r] & B \arrow[r] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & {} \\
 & {} & {} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, dash] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d, dash] & \cdot \arrow[r, dash] \arrow[d, dash] & {} \\
 & {} & {} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d, dash] & A \arrow[r] \arrow[d, dash] & 0 \\
 & {} & {} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r, dash] \arrow[d] & {} \\
{} \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r, dash] \arrow[d] & {} \\
 & 0 & 0 &
\end{tikzcd}
\tag{12}
$$

Proof. We will prove the isomorphism statements for the first diagram; the proofs for the remaining diagrams are obtained by reversing the roles of rows and columns, and/or reversing the directions of all arrows.

In the first diagram, Corollary 2.1, applied to the arrow $0 \to B$, gives ${}^{\square}B \cong 0_{\square} = 0$. The exact sequence (10) associated with the map $0 \to A$ therefore ends $0 \to {}^{\square}A \to A\!- \to 0$, while the exact sequence (11) associated with the map $A \to B$ begins $0 \to A\!\mid \to A_{\square} \to 0$. This gives the two desired isomorphisms. ∎

Remarks. Corollaries 2.1 and 2.2 are easy to prove directly, so in these two proofs, the Salamander Lemma was not a necessary tool, but a guiding principle.

One should not be misled by apparent further dualizations. For instance, one might think that if in the leftmost diagram of (12), one assumed row-exactness at A rather than at B, one would get isomorphisms ${}^{\square}B \cong B\!-$ and $B\!\mid \cong B_{\square}$ by reversing the directions of vertical arrows, and applying the result proved. However, there is no principle that allows us to reverse some arrows (the vertical ones) without reversing others (the horizontal arrows); and in fact, the asserted isomorphisms fail. E.g., if the vertical arrow down from

B is the identity map of a nonzero object, and all other objects of the double complex are zero, then row-exactness does hold at A (since the row containing A consists of zero-objects), but the maps ${}^{\square}B \to B\,\text{-}\!\bullet\!\text{-}$ and $B\,\vert\!\!\bullet \to B_{\square}$ both have the form $0 \to B$. (And if one conjectures instead that one or both of the other two intramural maps at B should be isomorphisms, the diagram with the horizontal map out of B an identity map and all other objects 0 belies these statements.) The common feature of the four situations of (12) that is not shared by the results of reversing only vertical or only horizontal arrows is that the arrow connecting A with B, and the arrow connecting a zero object with B, have the same orientation relative to B; i.e., either both go into it, or both come out of it.

Most of the "small" diagram-chasing lemmas of homological algebra can be obtained from the above two corollaries. For example:

2.3. Lemma. [The Sharp 3×3 Lemma.] *In the diagram below* (excluding*, respectively* including *the parenthesized arrows; and ignoring for now the boxes and dotted lines, which belong to the proof*)*, if all columns, and all rows but the first, are exact, then the first row* (*again excluding, respectively including the parenthesized arrow*) *is also exact.*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d] & (\longrightarrow 0) \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] & (\longrightarrow 0) \\
0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] & C'' & \\
 & (0) & & &
\end{tikzcd}
\tag{13}
$$

Proof. We first note that in view of the exactness of the columns, the first row of (13) consists of subobjects of the objects of the second row, and restrictions of the maps among these, hence it is, at least, a complex, so the whole diagram is a double complex.

Corollaries 2.1 and 2.2, combined with the exactness hypotheses not involving the parenthesized arrows, now give us

$$
\begin{aligned}
&A'\,\text{-}\!\bullet\!\text{-} \;\cong\; A'_{\square} \;\cong\; A'\,\vert\!\!\bullet \;=\; 0,\\
&B'\,\text{-}\!\bullet\!\text{-} \;\cong\; B'_{\square} \;\cong\; {}^{\square}B \;\cong\; A_{\square} \;\cong\; A\,\vert\!\!\bullet \;=\; 0 \qquad \text{(short dotted path in (13))},
\end{aligned}
\tag{14}
$$

and, assuming also the exactness conditions involving the parenthesized arrows,

$$
\begin{aligned}
&C'\,\text{-}\!\bullet\!\text{-} \;\cong\; C'_{\square} \;\cong\; {}^{\square}C \;\cong\; B_{\square} \;\cong\; {}^{\square}B'' \;\cong\; A''_{\square} \;\cong\; A''\,\vert\!\!\bullet \;=\; 0\\
&\text{(long dotted path in (13))}.
\end{aligned}
\tag{15}
$$

(As an example of how to determine which statement of which of those corollaries to use in each case, consider the first isomorphism of the first line of (14). Since it concerns an intramural isomorphism, it must be an application of Corollary 2.2. Since the one of

${}^{\square}A'$, $A'_{\square}$ that it involves faces away from the zeroes of the diagram, it must come from the second row of isomorphisms in that corollary; and given that fact, since it involves a horizontal homology object, it must call on exactness at an object horizontally displaced from A', hence must come from the second or fourth diagram of (12). Looking at the placement of the zeroes, we see that it must come from the second diagram, and that the needed hypothesis, vertical exactness at the object to the right of A', is indeed present. The isomorphisms other than the first and the last in each line of (14) and in (15), corresponding to extramural maps, follow from Corollary 2.1.)

The consequent triviality of the two (respectively three) horizontal homology objects with which (14) (and (15)) begin gives the desired exactness of the top row of (13). ∎

The diagonal chains of donors and receptors which we followed in the above proof fulfill the promise that "long" connections would be reduced to composites of "short" ones. The proof of the next lemma continues this theme.

2.4. Lemma. [Snake Lemma, [1, p. 23], [4], [10, p. 158], [12, p. 50].] *If, in the commuting diagram at left below, both rows are exact, and we append a row of kernels and a row of cokernels to the vertical maps, as in the diagram at right,*

$$\begin{tikzcd} & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 & \end{tikzcd} \qquad \begin{tikzcd} & K_1 \arrow[r] \arrow[d] & K_2 \arrow[r] \arrow[d] & K_3 \arrow[d] & \\ & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[d] & \\ & C_1 \arrow[r] & C_2 \arrow[r] & C_3 & \end{tikzcd} \tag{16}$$

then those two rows fit together into an exact sequence

$$K_1 \longrightarrow K_2 \longrightarrow K_3 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow C_3. \tag{17}$$

Proof. We extend the right diagram of (16) to a double complex by attaching a kernel X_0 to the second row and a cokernel Y_4 to the third, and filling in zeroes everywhere else. In this complex, the three columns shown in (16) are exact, and we have horizontal exactness at X_1, X_2, X_3, Y_1, Y_2 and Y_3.

The exactness of (17) at K_2, i.e., the triviality of $K_2{\leftarrow}$, now follows from the following isomorphisms (the first a case of Corollary 2.2, the next four of Corollary 2.1; cf. second equation of (14)):

$$K_2{\leftarrow} \;\cong\; K_{2\,\square} \;\cong\; {}^{\square}X_2 \;\cong\; X_{1\,\square} \;\cong\; {}^{\square}Y_1 \;\cong\; 0_{\square} \;=\; 0. \tag{18}$$

Exactness at C_2 is shown similarly.

We now want to find a connecting map $K_3 \to C_1$ making (17) exact at these two objects. This is equivalent to an isomorphism between $\mathrm{Cok}(K_2 \to K_3) = K_3{\leftarrow}$ and

$\mathrm{Ker}(C_1 \to C_2) = C_{1\to\bullet-}$. And indeed, such an isomorphism is given by the composite

$$K_{3\to\bullet-} \cong K_{3\,\square} \cong {}^{\square}X_3 \cong X_{2\,\square} \cong {}^{\square}Y_2 \cong Y_{1\,\square} \cong {}^{\square}C_1 \cong C_{1\to\bullet-} \tag{19}$$

of two intramural and five extramural maps shown as the dotted path in (16), which are again isomorphisms by Corollaries 2.2 and 2.1. ∎

The next result, whose proof by the same method we leave to the reader, establishes isomorphisms between infinitely many pairs of homology objects in a double complex bordered by zeroes, either in two parallel, or two perpendicular directions. Before stating it, we need to make some choices about indexing.

2.5. Convention. *When the objects of a double complex are indexed by numerical subscripts, the first subscript will specify the row and the second the column, and these will increase downwards, respectively, to the right* (*as in the numbering of the entries of a matrix; but* not *as in the standard coordinatization of the* (x, y)*-plane*)*.*

Since our arrows also point downward and to the right, our complexes will be double *cochain* complexes; i.e., the boundary morphisms will go from lower- to higher-indexed objects. However, we will continue to call the constructed objects "homology objects", rather than "cohomology objects".

Here is the promised result, which the reader can easily prove by the method used for Lemmas 2.3 and 2.4.

2.6. Lemma. *If in the left-hand complex below, all rows but the first row shown* (*the row of* $A_{0,r}$*'s*) *and all columns but the first column shown* (*the column of* $A_{r,0}$*'s*)*, are exact, then the homologies of the first row and the first column are isomorphic:* $A_{0,r\to\bullet-} \cong A_{r,0\,\downarrow\bullet}$*.* (*And analogously for a complex bordered by zeroes on the* bottom *and the* right*.*)

If in the right-hand commuting diagram below, all columns are exact, and all rows but the first and last are exact, then the homologies of those two rows agree with a shift of $n-m-1$ *:* $A_{m,r\to\bullet-} \cong A_{n,r-n+m+1\to\bullet-}$*.* (*And analogously for a complex bordered on the left and right by zeroes.*)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_{00} \arrow[r] \arrow[d] & A_{01} \arrow[r] \arrow[d] & A_{02} \arrow[r] \arrow[d] & A_{03} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & A_{10} \arrow[r] \arrow[d] & A_{11} \arrow[r] \arrow[d] & A_{12} \arrow[r] \arrow[d] & A_{13} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & A_{20} \arrow[r] \arrow[d] & A_{21} \arrow[r] \arrow[d] & A_{22} \arrow[r] \arrow[d] & A_{23} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & A_{30} \arrow[r] \arrow[d] & A_{31} \arrow[r] \arrow[d] & A_{32} \arrow[r] \arrow[d] & A_{33} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & A_{m,-1} \arrow[r] \arrow[d] & A_{m,0} \arrow[r] \arrow[d] & A_{m,1} \arrow[r] \arrow[d] & A_{m,2} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & {} \\
 & \vdots & \vdots & \vdots & \vdots & \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A_{n,-1} \arrow[r] \arrow[d] & A_{n,0} \arrow[r] \arrow[d] & A_{n,1} \arrow[r] \arrow[d] & A_{n,2} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
\tag{20}$$

∎

In contrast to the first of these results, if we form a double complex bordered on the *top* and the *right* (or on the *bottom* and the *left*) by zeroes, and we again assume all rows and columns exact except those adjacent to the indicated row and column of zeroes, there will in general be no relation between their homologies. For a counterexample, one can take a double complex in which all objects are zero except for a "staircase" of isomorphic objects running upward to the right till it hits one of the "borders". The place where it hits that border will be the only place where a nonzero homology object occurs. (If one tries to construct a similar counterexample to the first assertion of the above lemma by running a staircase of isomorphisms upward to the *left*, one finds that the resulting array of objects and morphisms is not a commutative diagram.)

3. Weakly bounded double complexes

Before exploring uses of the full statement of the Salamander Lemma, it will be instructive to consider a mild generalization of our last result. Suppose that as in the right-hand diagram of (20) we have a double complex with exact columns, bounded above the m-th row and below the n-th row by zeros. But rather than assuming exactness in all but the m-th and n-th rows, let us assume it in all rows but the i-th and j-th, for some i and j with $m \leq i < j \leq n$. I claim it will still be true that the homologies of these rows agree up to a shift:

$$A_{i,r}{\text{-}\!\bullet\!\text{-}} \;\cong\; A_{j,r-j+i+1}{\text{-}\!\bullet\!\text{-}}. \tag{21}$$

Indeed, first note that by composing extramural isomorphisms as in the preceding section, we get

$$A_{i,r\,{}_\square} \;\cong\; {}^\square\! A_{j,r-j+i+1}. \tag{22}$$

So the problem is to strengthen Corollary 2.2 to show that the objects of (22) are isomorphic (by the intramural maps) to their counterparts in (21). The required generalization of Corollary 2.2 is quite simple.

3.1. Corollary. *Suppose A is an object of a double complex, and the nearby donor and receptor objects marked "$\circ$" in one of the diagrams below are zero.*

$$\begin{tikzcd} & \bullet \arrow[d, "\circ"] & \\ \bullet \arrow[r] & A \arrow[r, "\circ"] \arrow[d] & \bullet \\ & \bullet & \end{tikzcd} \qquad\qquad \begin{tikzcd} & \bullet \arrow[d] & \\ \bullet \arrow[r, "\circ"'] & A \arrow[r] \arrow[d, "\circ"'] & \bullet \\ & \bullet & \end{tikzcd} \tag{23}$$

$$\begin{array}{c} {}^\square\! A \cong A{\overset{|}{\underset{|}{\bullet}}} \\ A{\text{-}\!\bullet\!\text{-}} \cong A_\square \end{array} \qquad\qquad \begin{array}{c} {}^\square\! A \cong A{\text{-}\!\bullet\!\text{-}} \\ A{\overset{|}{\underset{|}{\bullet}}} \cong A_\square \end{array}$$

Then the two intramural isomorphisms indicated below that diagram hold.

Proof. To get the first isomorphism of the first diagram, apply the Salamander Lemma to the arrow coming vertically into A; to get the second, apply it to the arrow coming horizontally out of A. In the second diagram, similarly apply it to the two arrows bearing the "$\circ$"s. ∎

Now in the situation of the first paragraph of this section, our double complex is exact horizontally above the i-th row, and vertically everywhere, so we can use Corollary 2.1 to connect any donor above the i-th row, or any receptor at or above that row, to a donor or receptor above the m-th row, proving it zero. Corollary 3.1 then shows the left-hand side of (21) to be isomorphic to the left-hand side of (22). Similarly, using exactness below the j-th row, and vanishing below the n-th, we find that the right-hand sides of those displays are isomorphic. Thus, (22) yields (21), as desired.

What if we have a double complex in which all columns, and all but the i-th and j-th rows, are exact, but we do not assume that all but finitely many rows are zero? Starting from the receptor at any object of the i-th row, we can still get an infinite chain of isomorphisms going upward and to the right:

$$ {}^{\square}A_{i,s} \;\cong\; A_{i-1,s}{}_{\square} \;\cong\; {}^{\square}A_{i-1,s+1} \;\cong\; A_{i-2,s+1}{}_{\square} \;\cong\; \ldots\,, \tag{24}$$

but we can no longer assert that the common value is zero; and similarly below the j-th row. However, there are certainly weaker hypotheses than the one we were using above that *will* allow us to say this common value is 0; e.g., the existence of zero quadrants (rather than half-planes) on the upper right and lower left. Let us make, still more generally,

3.2. Definition. *A double complex* $(A_{r,s})$ *will be called* weakly bounded *if for every* r *and* s, *there exists a* positive *integer* n *such that* ${}^{\square}A_{r-n,s+n}$ *or* $A_{r-n-1,s+n}{}_{\square}$ *is zero, and also a* negative *integer* n *with the same property.*

The above discussion now yields the first statement of the next corollary; the final statement is seen to hold by a similar argument.

3.3. Corollary. [to proof of Lemma 2.6.] *Let* $(A_{r,s})$ *be a* weakly bounded *double complex.*

If all columns are exact, and all rows but the i*-th and* j*-th are exact, where* $i < j$*, then the homologies of these rows are isomorphic with a shift:* $A_{i,r}{}^{\text{-}\bullet\text{-}} \cong A_{j,r-j+i+1}{}^{\text{-}\bullet\text{-}}$*. The analogous statement holds if all rows and all but two columns are exact.*

If all rows but the i*-th, and all columns but the* j*-th are exact* (i *and* j *arbitrary*)*, then the* i*-th row and* j*-th column have isomorphic homologies:* $A_{i,r}{}^{\text{-}\bullet\text{-}} \cong A_{r-j+i,j}{}^{\vert\!\bullet}$. ∎

To see that the above corollary fails without the hypothesis of weak boundedness, consider again a double complex that is zero except for a "staircase" of copies of a nonzero object and identity maps between them:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d, equal] & \\
\arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r, equal] \arrow[d, equal] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
\arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} & {} & {} &
\end{tikzcd} \tag{25}$$

All rows and all columns are then exact except the *row* containing the lowest "A". Considering that row and any other row, we get a contradiction to the first conclusion of Corollary 3.3. Considering that row and any column gives a contradiction to the final conclusion.

We remark that if, in Corollary 3.3, we make the substitution $s = r + i$ in the subscripts, then our isomorphisms take the forms $A_{i,s-i}{\rightarrow} \cong A_{j,s-j+1}{\rightarrow}$ and $A_{i,s-i}{\rightarrow} \cong A_{s-j,j}{\downarrow}$. These formulas are more symmetric than those using r, but I find them a little less easy to think about, because the variable index s never appears alone. But in later results, Lemmas 4.4 and 5.1, where the analog of the r-indexing would be messier than it is here, we shall use the analog of this s-indexing.

4. Long exact sequences

At this point it would be easy to apply Lemma 1.7 and Corollaries 2.1 and 2.2 to give a quick construction of the long exact sequence of homologies associated with a short exact sequences of complexes; the reader may wish to do so for him or her self. But we shall find it more instructive to examine how the six-term exact "salamander" sequences we have associated with the arrows of a double complex link together under various weaker hypotheses, and see that the above long exact sequence is the simplest interesting case of some more general phenomena.

Let B be any object of a double complex, with some neighboring objects labeled as follows.

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & E \arrow[d] & \\
A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[d] \\
 & F \arrow[r] & G\,,
\end{tikzcd} \tag{26}$$

and let us consider the six-term exact sequences associated with the four arrows into and out of B. These piece together as in the following diagram, where the central square and each of the four triangular wedges commute:

$$\begin{tikzcd}
 & & & D_{\Box} \arrow[d] & & & & \\
 & & & E{\uparrow} \arrow[d] & & & & \\
 & & & E_{\Box} \arrow[d] \arrow[dr, bend left] & & & & \\
D_{\Box} \arrow[r] & A{\leftarrow} \arrow[r] & A_{\Box} \arrow[r] \arrow[dr, bend right] & {}^{\Box}B \arrow[r] \arrow[d] & B{\leftarrow} \arrow[d] \arrow[dr, bend left] & & & \\
 & & & B{\uparrow} \arrow[r] \arrow[dr, bend right] & B_{\Box} \arrow[r] \arrow[d] & {}^{\Box}F \arrow[r] & F{\uparrow} \arrow[r] & {}^{\Box}G \\
 & & & & {}^{\Box}C \arrow[d] & & & \\
 & & & & C{\leftarrow} \arrow[d] & & & \\
 & & & & {}^{\Box}G & & &
\end{tikzcd} \tag{27}$$

We now note what happens if (26) is vertically or horizontally exact at B.

4.1. LEMMA. *Suppose in* (26) *that* $B{\uparrow} = 0$, *or that* $B{\leftarrow} = 0$, *or more generally, that the intramural map* ${}^{\Box}B \to B_{\Box}$ *is zero. Then the following two* 9*-term sequences* (*the first* *obtained from the "left-hand" and "bottom" branches of* (27)*, the second from the "top" and "right-hand" branches*) *are exact:*

$$D_{\square} \longrightarrow A{\text{-}\!\bullet\!\text{-}} \longrightarrow A_{\square} \longrightarrow {}^{\square}B \longrightarrow B{\text{-}\!\bullet\!\text{-}} \longrightarrow B_{\square} \longrightarrow {}^{\square}C \longrightarrow C{\text{-}\!\bullet\!\text{-}} \longrightarrow {}^{\square}G, \tag{28}$$

$$D_{\square} \longrightarrow E{\shortmid\!\bullet} \longrightarrow E_{\square} \longrightarrow {}^{\square}B \longrightarrow B{\shortmid\!\bullet} \longrightarrow B_{\square} \longrightarrow {}^{\square}F \longrightarrow F{\shortmid\!\bullet} \longrightarrow {}^{\square}G. \tag{29}$$

Proof. The exactness of the 6-term sequences of which (27) is composed gives the exactness of (28) and (29) everywhere but at the middle terms, $B{\text{-}\!\bullet\!\text{-}}$ and $B{\shortmid\!\bullet}$. That the composite map through that middle term equals zero is, in each case, our hypothesis on the intramural map ${}^{\square}B \to B_{\square}$. That, conversely, the kernel of the map out of that middle object is contained in the image of the map going into it can be seen from (27); e.g., in the case of (28), we see from (27) that the kernel of the map $B{\text{-}\!\bullet\!\text{-}} \to B_{\square}$ is the image of the map $E_{\square} \to B{\text{-}\!\bullet\!\text{-}}$, and that map factors through the map ${}^{\square}B \to B{\text{-}\!\bullet\!\text{-}}$. (This is the commutativity of the topmost triangular wedge in (27). A dual proof can be gotten using the right-hand wedge.) ∎

In noting applications of the above result, we shall, for brevity, restrict ourselves to (28); the corresponding consequences of (29) follow by symmetry. Of the alternative hypotheses of Lemma 4.1, the condition $B{\text{-}\!\bullet\!\text{-}} = 0$ makes (28) degenerate, while the more general statement that ${}^{\square}B \to B_{\square}$ is zero does not correspond to any condition in the standard language of homological algebra; so in the following corollary, we focus mainly on the condition $B{\shortmid\!\bullet} = 0$.

4.2. Corollary. *If in a double complex, a piece of which is labeled as in* (26)*, the* vertical *homologies are zero for all objects in the* row $\cdots \to A \to B \to C \to \ldots$ (*or more generally, if the intramural map from receptor to donor is zero for each object of that row*)*, then the following sequence of horizontal homology objects, donors and receptors, and intramural and extramural maps, is exact:*

$$\cdots \longrightarrow {}^{\square}A \longrightarrow A{\text{-}\!\bullet\!\text{-}} \longrightarrow A_{\square} \longrightarrow {}^{\square}B \longrightarrow B{\text{-}\!\bullet\!\text{-}} \longrightarrow B_{\square} \longrightarrow {}^{\square}C \longrightarrow C{\text{-}\!\bullet\!\text{-}} \longrightarrow C_{\square} \longrightarrow \cdots . \tag{30}$$

Proof. At each object of the indicated row of (26), write down the exact sequence corresponding to (28), leaving off the first and last terms. The remaining parts of these sequences overlap, giving (30). ∎

When the vertical homology in our double complex is *everywhere* zero, the exact sequences (30) arising from successive rows are linked, at every third position, by isomorphisms given by Corollary 2.1, as described in

4.3. Corollary. *If in a double complex*

$$
\begin{tikzcd}
& & & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
& & {} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & {} \\
& {} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & {} & \\
{} \arrow[r] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & {} & & \\
{} \arrow[r] & W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & {} & & \\
& {} & {} & {} & {} & & &
\end{tikzcd}
\tag{31}
$$

all columns *are exact, then the* rows *induce long exact sequences, which are linked by isomorphisms:*

$$
\begin{tikzcd}[column sep=small]
& {} \arrow[d, equal] & & & {} \arrow[d, equal] & & & {} \arrow[d, equal] & & & {} \arrow[d, equal] & \\
{} \arrow[r] & {}^{\square}A \arrow[r] & A\bullet \arrow[r] & A_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}B \arrow[r] & B\bullet \arrow[r] & B_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}C \arrow[r] & C\bullet \arrow[r] & C_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}D \arrow[r] & {} \\
{} \arrow[r] & K\bullet \arrow[r] & K_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}L \arrow[r] & L\bullet \arrow[r] & L_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}M \arrow[r] & M\bullet \arrow[r] & M_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}N \arrow[r] & N\bullet \arrow[r] & {} \\
{} \arrow[r] & P_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}Q \arrow[r] & Q\bullet \arrow[r] & Q_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}R \arrow[r] & R\bullet \arrow[r] & R_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}S \arrow[r] & S\bullet \arrow[r] & S_{\square} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & {}^{\square}W \arrow[r] & W\bullet \arrow[r] & W_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}X \arrow[r] & X\bullet \arrow[r] & X_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}Y \arrow[r] & Y\bullet \arrow[r] & Y_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}Z \arrow[r] & {} \\
& & & {} & & & {} & & & {} & &
\end{tikzcd}
\tag{32}
$$

∎

Note that in these long exact sequences, the classical homology objects form every third term – the terms of (32) that are not connected either above or below by isomorphisms.

Suppose now that in (31), in addition to all columns being exact, some row is exact. This means that in the system of long exact sequences (32), the corresponding row will have every third term zero; and so the maps connecting the remaining terms will be isomorphisms:

$$
\begin{tikzcd}[column sep=small]
& {} \arrow[d, equal] & & & {} \arrow[d, equal] & & & {} \arrow[d, equal] & & & {} \arrow[d, equal] & \\
{} \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & {} \\
& & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet & & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet & & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet & & \\
{} \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r] & {} \\
& & & {} & & & {} & & & {} & &
\end{tikzcd}
\tag{33}
$$

We see that these, together with the vertical isomorphisms, tie together the *preceding* and *following* exact sequence to give a system essentially like (32), except for a horizontal shift by one step. If n successive rows of the double complex are exact, we get a similar diagram with a shift by n steps.

If all rows are exact above a certain point, then we get infinite chains of isomorphisms going upward and to the right. If the complex is also weakly bounded (Definition 3.2), the common value along those chains will be zero; hence every third term of the long exact sequence corresponding to the top *nonexact* row of our double complex will be zero, so we again have isomorphisms between pairs of remaining terms; though not the same pairs as before: in each isomorphic pair, one of the members is now a horizontal homology object. If we regard classical homology objects as more interesting than donors and receptors, we may use these isomorphisms and the isomorphisms joining this row to the next to insert these homology objects in that row, in place of all the receptors.

For instance, if all rows of (31) above the top one shown are exact, and the complex is weakly bounded above, then in the top two rows of (32) we get

$$\begin{tikzcd} & & A\!\!-\!\!\bullet\!\!- \arrow[r, equal] & A_{\square} \arrow[d, equal] & & \\ {} \arrow[r] & K\!\!-\!\!\bullet\!\!- \arrow[r] & K_{\square} \arrow[r] \arrow[d, equal] & {}^{\square}L \arrow[r] & L\!\!-\!\!\bullet\!\!- \arrow[r] & {} \\ & & {} & & & \end{tikzcd},$$

which we can rewrite

$$\begin{tikzcd} {} \arrow[r] & K\!\!-\!\!\bullet\!\!- \arrow[r] & K_{\square} \arrow[r] \arrow[d, equal] & A\!\!-\!\!\bullet\!\!- \arrow[r] & L\!\!-\!\!\bullet\!\!- \arrow[r] & {} \\ & & {} & & & \end{tikzcd}.$$

Of course, if, say, the second row of (31) (unlike the first) happens to be exact, then the objects $K\!\!-\!\!\bullet\!\!-$, $L\!\!-\!\!\bullet\!\!-$ etc. in the above exact sequence are zero, allowing us to pull the homology objects from the first row down yet another step, and insert them into the long exact sequence arising from the third row. Continuing in this way as long as we find exact rows in (31), we get a linked system of long exact sequences, of which the top sequence has, as *two* out of every three terms, horizontal homology objects, and arises from the two highest non-exact rows of (31) (assuming there are at least two). The obvious analogous situation holds if, instead, all rows *below* some point are exact.

If our original double complex has only three nonexact rows, then we can see that, working in this way from both ends, we get a single long exact sequence with horizontal homology objects for all its terms:

4.4. Lemma. *Suppose we are given a weakly bounded double complex, with objects $A_{h,r}$, all columns exact, and all rows exact except the i-th, j-th and k-th, where $i < j < k$. Then we get a long exact sequence*

$$\cdots \to A_{i,\,s-i-1}\!\!-\!\!\bullet\!\!- \to A_{j,\,s-j}\!\!-\!\!\bullet\!\!- \to A_{k,\,s-k+1}\!\!-\!\!\bullet\!\!- \to A_{i,\,s-i}\!\!-\!\!\bullet\!\!- \to A_{j,\,s-j+1}\!\!-\!\!\bullet\!\!- \to A_{k,\,s-k+2}\!\!-\!\!\bullet\!\!- \to \cdots. \tag{34}$$

(*Regarding the indexing, cf. last paragraph of §3.*) ■

What if we have four rather than three non-exact rows (again in a weakly bounded double complex with exact columns)? Assuming for concreteness that our double complex is (31), and that all rows but the four shown there are exact, we find that (32) collapses to

$$\begin{tikzcd} \cdots \arrow[r] & K\!\!-\!\!\bullet\!\!- \arrow[r] & K_{\square} \arrow[r] \arrow[d, equal] & A\!\!-\!\!\bullet\!\!- \arrow[r] & L\!\!-\!\!\bullet\!\!- \arrow[r] & L_{\square} \arrow[r] \arrow[d, equal] & B\!\!-\!\!\bullet\!\!- \arrow[r] & M\!\!-\!\!\bullet\!\!- \arrow[r] & M_{\square} \arrow[r] \arrow[d, equal] & C\!\!-\!\!\bullet\!\!- \arrow[r] & N\!\!-\!\!\bullet\!\!- \arrow[r] & \cdots \\ \cdots \arrow[r] & W\!\!-\!\!\bullet\!\!- \arrow[r] & {}^{\square}Q \arrow[r] & Q\!\!-\!\!\bullet\!\!- \arrow[r] & X\!\!-\!\!\bullet\!\!- \arrow[r] & {}^{\square}R \arrow[r] & R\!\!-\!\!\bullet\!\!- \arrow[r] & Y\!\!-\!\!\bullet\!\!- \arrow[r] & {}^{\square}S \arrow[r] & S\!\!-\!\!\bullet\!\!- \arrow[r] & Z\!\!-\!\!\bullet\!\!- \arrow[r] & \cdots \end{tikzcd}. \tag{35}$$

5. Some rows, and some columns

We have just seen what happens when all columns, and all but a finite number of rows of a weakly bounded double complex are exact; the corresponding results hold, of course, when all rows and all but a finite number of columns are exact.

One can look, more generally, at the situation where

$$\text{All but } m \text{ rows, and all but } n \text{ columns are exact.} \tag{36}$$

In this section we shall examine the sort of behavior that this leads to. For the first result, I give a formal statement, Lemma 5.1, and a sketch of the proof, of which the reader can check the details, following the technique of the preceding section. The proofs of the remaining results discussed use the same ideas.

The case $m+n \leq 2$ of (36) is covered by Corollary 3.3 above. The case $m+n=3$ is covered (up to row-column reversal) by Lemma 4.4, together with

5.1. Lemma. *Suppose we have a weakly bounded double complex with objects $A_{h,r}$, all rows exact but the i-th and j-th, where $i<j$, and all columns exact but the k-th. Then we have a long exact sequence*

$$\cdots \to A_{i,s-i-1}{}^{\text{-}\!\bullet\!\text{-}} \to A_{j,s-j}{}^{\text{-}\!\bullet\!\text{-}} \to A_{s-k,k}{}^{\stackrel{|}{\bullet}} \to A_{i,s-i}{}^{\text{-}\!\bullet\!\text{-}} \to A_{j,s-j+1}{}^{\text{-}\!\bullet\!\text{-}} \to A_{s-k+1,k}{}^{\stackrel{|}{\bullet}} \to \cdots . \tag{37}$$

Sketch of Proof Write out the salamander exact sequences corresponding to the *horizontal* arrow out of $A_{i,r}$ for each $r<k$, to the *vertical* arrow out of $A_{h,k}$ for $h=i,\ldots,j-1$, and to the *horizontal* arrow out of $A_{j,r}$ for each $r \geq k$.

Except where we come to a corner, these exact sequences piece together (due to exactness of all other rows and columns) as in (30). When we do turn a corner, we get a different sort of piecing together; e.g., if we take the maps $A_{i,k-1} \to A_{i,k} \to A_{i+1,k}$ for the arrows $A \to B \to F$ of (26), then the $E_{\square}$ and ${}^{\square}C$ of (26) are both zero, due to weak boundedness, so that in (27), the two horizontal exact sequences collapse into one 8-term exact sequence. So the path of arrows in our double complex described in the preceding paragraph leads to a single long exact sequence of homology objects, donors, and receptors.

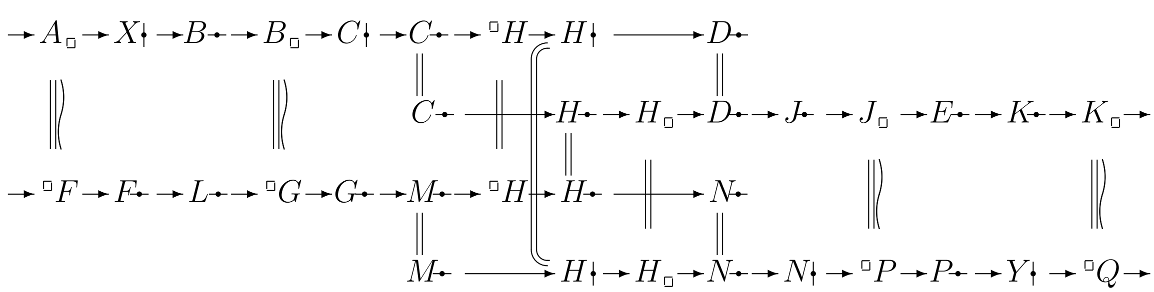
For each donor or receptor in this sequence, we now use a string of extramural isomorphisms (consequences of Corollary 2.1 and the exactness of all but our three exceptional rows and columns) to connect it with a receptor or donor at an object of one of the other two non-exact rows or columns. (In each case, there is only one direction we can go by extramural isomorphisms from our donor or receptor object, without crossing the non-exact row or column we are on, and this indeed leads to an object of another non-exact row or column.) We know from weak boundedness that the donor and receptor objects on the *other* side of the row or column we have arrived at are zero, and so conclude by Corollary 3.1 that the receptor or donor we have reached is isomorphic to a vertical or horizontal homology object in that row or column.

Thus, we get an exact sequence in which all objects are vertical or horizontal homology objects. ∎

So far, the general case of (36) has given results as nice as when m or n is 0. But now consider $m+n=4$. We saw in (35) what happens when $n=0$; let us compare this with the case of a double complex with three not necessarily exact rows and one not necessarily exact column, such as the following (where we have darkened the arrows in the not necessarily exact rows and columns).

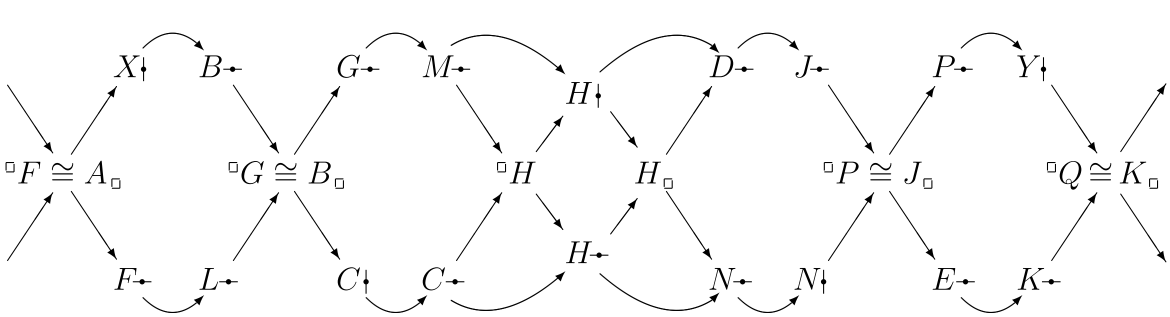
$$\begin{tikzcd}
{} & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & X \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & {} \\
{} \arrow[r] & A \arrow[r]\arrow[d] & B \arrow[r]\arrow[d] & C \arrow[r]\arrow[d] & D \arrow[r]\arrow[d] & E \arrow[r]\arrow[d] & {} \\
{} \arrow[r] & F \arrow[r]\arrow[d] & G \arrow[r]\arrow[d] & H \arrow[r]\arrow[d] & J \arrow[r]\arrow[d] & K \arrow[r]\arrow[d] & {} \\
{} \arrow[r] & L \arrow[r]\arrow[d] & M \arrow[r]\arrow[d] & N \arrow[r]\arrow[d] & P \arrow[r]\arrow[d] & Q \arrow[r]\arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & Y \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & \cdot \arrow[r]\arrow[d] & {} \\
{} & {} & {} & {} & {} & {} & {}
\end{tikzcd} \tag{38}$$

One finds that this double complex leads to a system of four linked "half-long" exact sequences. To the left and to the right, the diagram looks like the $n=0$ case, (35), but there is a peculiar "splicing" in the middle:



(39)

Here is the same diagram, redrawn more smoothly.



(40)

The exact sequences are those chains of arrows which can be followed without making sharp turns.

We remark that the first step in verifying the exactness of the sequences in (39), equivalently, (40) is to check that the following isomorphisms follow from Corollaries 2.1 and 3.1.

$$\begin{array}{lllll} X\dot{\vert} \cong {}^{\square}X \cong {}^{\square}B, & C\dot{\vert} \cong {}^{\square}C, & C\dot{-} \cong C_{\square}\,, & D\dot{-} \cong D_{\square} \cong {}^{\square}J\,, & {}^{\square}P \cong J_{\square}\,, \\ {}^{\square}Q \cong K_{\square}\,, & E\dot{-} \cong E_{\square} \cong {}^{\square}K, & & L\dot{-} \cong {}^{\square}L \cong F_{\square}\,, & {}^{\square}F \cong A_{\square}\,, \\ Y\dot{\vert} \cong Y_{\square} \cong P_{\square}, & N\dot{\vert} \cong N_{\square}, & N\dot{-} \cong {}^{\square}N, & M\dot{-} \cong {}^{\square}M \cong G_{\square}, & {}^{\square}G \cong B_{\square}\,. \end{array} \tag{41}$$

Using these, the verification of the exactness conditions is immediate. (Note that the part of (40) between " ${}^{\square}G \cong B_{\square}$ " and " ${}^{\square}P \cong J_{\square}$ " is, up to labeling, just a copy of (27), with the top and left nodes of (27) identified, and likewise the bottom and right nodes, and with substitutions from (41) made where appropriate. So all the exactness conditions in that part of the diagram are immediate.)

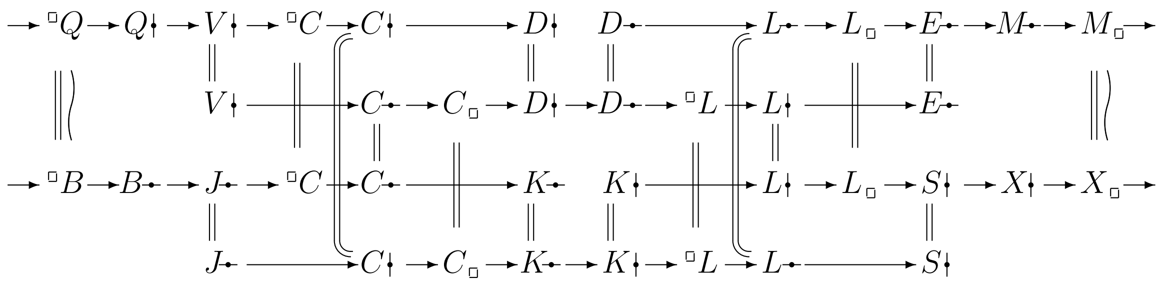
The interpolation of some exact rows between the three nonexact rows of (38) does not affect the resulting system of exact sequences (39), (40) except by a shift of indices.

General values of m and n in (36) yield systems that, for most of their length, consist of $m+n-2$ intertwining exact sequences (cf. (33)), but have finitely many "splicings"; essentially, one for each object of the given double complex which lies at the intersection of a nonexact row and a nonexact column, and does *not* have, either to its upper right or lower left, a region where (due to exactness and weak boundedness) all donors and receptors are zero. Thus, the one splice in (39) and (40) comes from the object H of (38).

If $m=n=2,$ as in

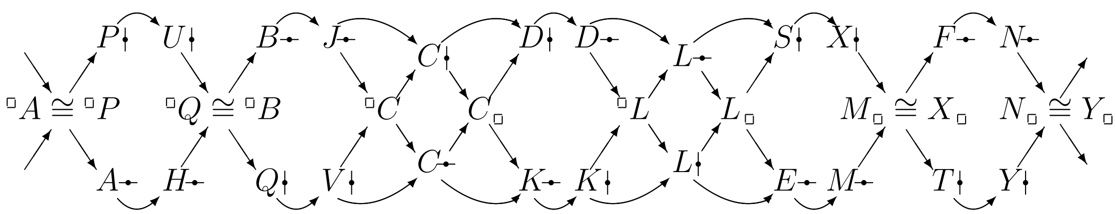
$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
& {} & {} & {} & {} & {} & {} &
\end{tikzcd} \tag{42}$$

then the two objects C and L lead to two "splicings":



(43)

The same diagram in "smooth" format (and carried one step further at each end) is



(44)

If exact rows or columns are introduced between the given nonexact ones, the splicings move farther apart (as the "staircases" on which C and L lie move apart), with a "normal" stretch between them.

6. Total homology

It is probably foolhardy, at very least, for someone who does not know the theory of spectral sequences to attempt to say something about the total homology of a double complex. However, I shall note here some connections between that subject and the constructions $^{\square}A$ and $A_{\square}$ we have been working with.

Let us be given

$$\begin{array}{l}\text{a double complex with objects } A_{i,r}, \text{ vertical arrows } \delta_1 : A_{i,r} \to A_{i+1,r}, \\ \text{and horizontal arrows } \delta_2 : A_{i,r} \to A_{i,r+1}, \ \ (i, r \in \mathbb{Z}).\end{array} \tag{45}$$

In particular, at each object $A_{i,r}$, we have

$$\delta_1\delta_2 \ = \ \delta_2\delta_1, \qquad \delta_1\delta_1 \ = \ 0, \qquad \delta_2\delta_2 \ = \ 0. \tag{46}$$

At this point, one usually defines the *total complex* induced by this double complex to have for objects the direct sums $A_n = \bigoplus_{i+r=n} A_{i,r}$, assuming the countable direct sum construction to be defined and exact in our abelian category $\mathbf{A}$. But we may as well be more general. Let $\mathbf{A}^{\mathbb{Z}}$ denote the abelian category of all $\mathbb{Z}$-tuples $X = (X_i)_{i \in \mathbb{Z}}$ of objects of $\mathbf{A}$, and let $\sum : \mathbf{A}^{\mathbb{Z}} \to \mathbf{B}$ be *any* exact functor into an abelian category $\mathbf{B}$

which commutes with shift, i.e., has the property that for $(X_i) \in \mathbf{A}^{\mathbb{Z}}$ we have a functorial isomorphism

$$\textstyle\sum_i X_i \;\cong\; \sum_i X_{i+1}. \tag{47}$$

For instance, suppose $\mathbf{A} = \mathbf{B} =$ the category of all R-modules for R a fixed ring. (As noted in §1, "module" can mean either left or right module.) Then we might take $\sum$ to be: (i) the operator of direct sum, or (ii) the operator of direct product, or (iii) or (iv) the right- or left-truncated product operators, given by $\sum X_i = (\prod_{i<0} X_i) \times (\bigoplus_{i\geq 0} X_i)$, respectively, $\sum X_i = (\bigoplus_{i<0} X_i) \times (\prod_{i\geq 0} X_i)$ (these might be called "formal Laurent sum" operations; the reader should check that they indeed satisfy (47)), or even some very "un-sum-like" constructions, such as (v) $\sum X_i = (\prod X_i)/(\bigoplus X_i)$, or more generally (vi) the reduced product of the R-modules X_i $(i \in \mathbb{Z})$ with respect to any translation-invariant filter on $\mathbb{Z}$.

Before saying what we will do with these functors, let me digress and point out that in any abelian category $\mathbf{A}$ with countable coproducts, i.e., countable direct sums, the functor $\bigoplus_{i\in\mathbb{Z}} : \mathbf{A}^{\mathbb{Z}} \to \mathbf{A}$ satisfies (47) and is *right* exact (since it is a left adjoint, and therefore respects coequalizers); and, dually, when the countable direct product functor $\prod_{i\in\mathbb{Z}}$ exists, it satisfies (47) and is left exact. For $\mathbf{A}$ the category of all R-modules, it is easy to check that these constructions are exact on both sides; but there are abelian categories in which countable products or coproducts exist but are not exact. For instance, in the category $\mathbf{A}$ of *torsion* abelian groups, infinite products are given by the torsion subgroup of the direct product as groups [10, Exercise I.8]. It is easy to see that the direct product in this category of the family of short exact sequences $0 \to \mathbb{Z}/(p^i) \to \mathbb{Z}/(p^{i+1}) \to \mathbb{Z}/(p) \to 0$, for p a fixed prime and i ranging over the natural numbers, loses exactness on the right: no element of the product of the middle terms maps to the element $(1, 1, 1, \dots)$ of the product of the right-hand terms. (As described, this example is a family of short exact sequences indexed by $\mathbb{N}$; but by associating copies of the zero short exact sequence to indices $i < 0$, we get an example of non-right-exactness of products indexed by $\mathbb{Z}$.) Applying Pontryagin duality [14, Theorem 1.7.2] to this example, we get non-left-exactness of countable *coproducts* in the category of totally disconnected compact Hausdorff abelian groups (though it seems harder to describe the elements involved). Thus, these two functors, and others like them, are *excluded* as candidates for the $\sum$ we are considering.

Returning to where we left off, suppose we are given an exact functor $\sum : \mathbf{A}^{\mathbb{Z}} \to \mathbf{B}$ satisfying (47), and a double complex (45) in $\mathbf{A}$. Then we define

$$A_n \;=\; \textstyle\sum_{i\in\mathbb{Z}} \; A_{i,n-i}, \quad \text{for each } n \in \mathbb{Z}. \tag{48}$$

The families of maps δ_1 and δ_2 of (45) induce maps which we shall denote by the same symbols,

$$\delta_1, \delta_2 : \; A_n \;\to\; A_{n+1}, \quad \text{for each } n \in \mathbb{Z}. \tag{49}$$

(Remark: the isomorphism (47) is needed in the definition of δ_1, but not in that of δ_2; essentially because we decided arbitrarily that the i of the operator $\sum_i$ in (48) would index the *first* subscript of $A_{i,n-i}$.) These maps will again clearly satisfy (46). Since δ_1 and δ_2 now represent maps which can simultaneously have the same range and the same domain (as in (49)), we can add and subtract them, and (46) immediately yields

$$(\delta_1+\delta_2)\,(\delta_1-\delta_2) \;=\; 0 \;=\; (\delta_1-\delta_2)\,(\delta_1+\delta_2). \tag{50}$$

Thus, if for each n we let

$$\delta \;=\; \delta_2+(-1)^n\delta_1 :\; A_n \longrightarrow A_{n+1}, \tag{51}$$

we get a complex

$$\cdots \xrightarrow{\ \delta\ } A_{n-1} \xrightarrow{\ \delta\ } A_n \xrightarrow{\ \delta\ } A_{n+1} \xrightarrow{\ \delta\ } \cdots . \tag{52}$$

We shall call (52) the *total complex* (with respect to the functor $\sum$) of our given double complex. Since the maps δ come from maps going downward and to the right on our original double complex, we shall denote the homology objects of the above complex by

$$A_{n\searrow} \;=\; \mathrm{Ker}(A_n \xrightarrow{\delta} A_{n+1})\,/\,\mathrm{Im}(A_{n-1} \xrightarrow{\delta} A_n). \tag{53}$$

So far, this is nothing new. We now bring our donor and receptor objects into the picture. Let us define

$$A_{n\square} = \textstyle\sum_i\ A_{i,n-i\,\square}\,, \qquad {}^{\square}A_n = \textstyle\sum_i\ {}^{\square}A_{i,n-i}\,. \tag{54}$$

From the exactness and shift-invariance of $\sum$, it follows that within each object A_n, subobjects such as $\mathrm{Ker}(\delta_1\colon A_n \to A_{n+1})$ and $\mathrm{Im}(\delta_1\colon A_{n-1} \to A_n)$ will be given by the corresponding "sums", $\sum_i \mathrm{Ker}(\delta_1\colon A_{i,n-i} \to A_{i+1,n-i})$ and $\sum_i \mathrm{Im}(\delta_1\colon A_{i-1,n-i} \to A_{i,n-i})$, and similarly for more complicated expressions. One deduces that

$$A_{n\square} \;=\; \mathrm{Ker}(\delta_1\delta_2)/(\mathrm{Im}\,\delta_1{+}\mathrm{Im}\,\delta_2), \quad {}^{\square}A_n = \;(\mathrm{Ker}\,\delta_1{\cap}\,\mathrm{Ker}\,\delta_2)/\mathrm{Im}(\delta_1\delta_2). \tag{55}$$

(Here, in the numerators, the symbols δ_1, δ_2, $\delta_1\delta_2$, denote the maps so named having domain A_n, and in the denominators, the maps with codomain A_n.)

In view of (55), the identity morphism of A_n induces intramural maps

$$^{\square}A_n \longrightarrow A_{n\searrow} \longrightarrow A_{n\square}\,. \tag{56}$$

(We could have defined $A_{n\downarrow}$ and $A_{n\rightarrow}$ analogously to (54), noted characterizations of them analogous to (55), and gotten a commuting diagram

$$\begin{tikzcd}
 & {}^{\square}A_n \arrow[dl] \arrow[d] \arrow[dr] & \\
A_{n\downarrow} \arrow[dr] & A_{n\searrow} \arrow[d] & A_{n\rightarrow}\,; \arrow[dl] \\
 & A_{n\square} &
\end{tikzcd} \tag{57}$$

but we shall not need these additional objects.)

Finally, the two sets of *extra*mural maps constructed from our original double complex in §1, combined with (54), yield, for each n, two maps which we shall call

$$A_{n\square} \overset{\overline{\delta}_1}{\underset{\overline{\delta}_2}{\rightrightarrows}} {}^{\square}A_{n+1}. \tag{58}$$

In terms of the description (55) of $A_{n\square}$ and ${}^{\square}A_{n+1}$, we see that $\overline{\delta}_1$ and $\overline{\delta}_2$ are induced in $\mathbf{B}$ by $\delta_1,\ \delta_2 : A_n \to A_{n+1}$.

Let us now write (analogous to (51)),

$$\overline{\delta} \ = \ \overline{\delta}_2 + (-1)^n\,\overline{\delta}_1 : \ A_{n\square} \longrightarrow {}^{\square}A_{n+1}. \tag{59}$$

We find that the composite of this map with the first intramural map of (56),

$$A_{n-1\square} \xrightarrow{\ \overline{\delta}\ } {}^{\square}A_n \longrightarrow A_n{}^{\searrow} \tag{60}$$

is zero, since the first arrow maps into the denominator of (53). This says that the two composites

$$A_{n-1\square} \overset{\overline{\delta}_1}{\underset{\overline{\delta}_2}{\rightrightarrows}} {}^{\square}A_n \longrightarrow A_n{}^{\searrow} \tag{61}$$

agree up to sign. Hence, below, we shall just refer to the composite involving $\overline{\delta}_1$. The same comments apply to the composites

$$A_n{}^{\searrow} \longrightarrow A_{n\square} \overset{\overline{\delta}_1}{\underset{\overline{\delta}_2}{\rightrightarrows}} {}^{\square}A_{n+1}. \tag{62}$$

We can now state a version of the Salamander Lemma for total homology.

6.1. Lemma. *In the above context, for each* n *the* 6*-term sequence of intramural and extramural maps and their composites*

$$A_{n-1\square} \xrightarrow[\text{cf. (61)}]{\overline{\delta}_1\ \ {}^{\square}A_n} A_n{}^{\searrow} \longrightarrow A_{n\square} \longrightarrow {}^{\square}A_{n+1} \longrightarrow A_{n+1}{}^{\searrow} \xrightarrow[\text{cf. (62)}]{A_{n+1\square}\ \ \overline{\delta}_1} {}^{\square}A_{n+2} \tag{63}$$

is exact.

Proof. Rather than re-proving this, let us deduce it from the Salamander Lemma (Lemma 1.7) by a trick. We define a double complex $(B_{i,r})$ in $\mathbf{B}$ in which $B_{i,r} = A_{i+r}$, the horizontal maps are the maps δ, and the vertical maps are the δ_1. Thus, letting $n = i + r$, we have

$$\begin{tikzcd}
 & B_{i-1,r} \arrow[d, "\delta_1"] & \arrow[d, "\delta_1"] & \\
 \arrow[r, "\delta"] & B_{i,r} \arrow[r, "\delta"] \arrow[d, "\delta_1"] & B_{i,r+1} \arrow[r, "\delta"] \arrow[d, "\delta_1"] & {} \\
 & {} & B_{i+1,r+1} &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A_{n-1} \arrow[d, "\delta_1"] & \arrow[d, "\delta_1"] & \\
 \arrow[r, "\delta_2 \mp \delta_1"] & A_n \arrow[r, "\delta_2 \pm \delta_1"] \arrow[d, "\delta_1"] & A_{n+1} \arrow[r, "\delta_2 \mp \delta_1"] \arrow[d, "\delta_1"] & {} \\
 & {} & A_{n+2} &
\end{tikzcd} \tag{64}$$

From (46) and (51), we see that $\delta_1\delta = \delta_1\delta_2 = \delta_2\delta_1 = \delta\delta_1$, and that in each object, $\operatorname{Im}\delta_1 + \operatorname{Im}\delta = \operatorname{Im}\delta_1 + \operatorname{Im}\delta_2$ and $\operatorname{Ker}\delta_1 \cap \operatorname{Ker}\delta = \operatorname{Ker}\delta_1 \cap \operatorname{Ker}\delta_2$. It easily follows that $B_{i,r\leftarrow} = A_{n\searrow}$, $B_{i,r\square} = A_{n\square}$, ${}^{\square}B_{i,r} = {}^{\square}A_n$, and that (63) is just the 6-term exact sequence associated with the above horizontal arrow of this double complex. ∎

Likewise, Corollary 4.2, applied to any row of the complex (64) gives:

6.2. Lemma. *If our original double complex has exact columns, then one has a long exact sequence in* $\mathbf{B}$*:*

$$\dots \xrightarrow{\overline{\delta}} {}^{\square}A_n \longrightarrow A_{n\searrow} \longrightarrow A_{n\square} \xrightarrow{\overline{\delta}} {}^{\square}A_{n+1} \longrightarrow \cdots . \qquad ∎ \tag{65}$$

(Note the curious property of this sequence: that the objects joined by the connecting morphisms $\overline{\delta}$ are *isomorphic* under a different map, $\overline{\delta}_1$, by Corollary 2.1.)

6.3. Corollary. *If our original double complex has* exact rows and columns*, and is* weakly bounded *(e.g., if $A_{i,r} = 0$ whenever i or r is negative), then all total homology objects $A_{n\searrow}$ are zero.*

Proof. In the original double complex $(A_{i,r})$, the donor and receptor objects form isomorphic chains going upward to the right and downward to the left by Corollary 2.1 and our exactness assumptions; hence, by weak boundedness, they are all zero. Thus, for all n, ${}^{\square}A_n = \sum {}^{\square}A_{i,n-i} = \sum 0 = 0$, and similarly $A_{n\square} = 0$. So by (65), $A_{n\searrow} = 0$. ∎

There is no more to be said about total homology under this hypothesis, so to finish this section, let us return to the weaker hypothesis of Lemma 6.2, and examine the behavior of the exact sequence (65) for various choices of $\mathbf{A}$ and $\sum$.

6.4. Convention. *For the remainder of this section, we shall assume as in Lemma 6.2 that our given double complex $(A_{i,r})_{i,r\in\mathbb{Z}}$ has exact columns.*

Consider now the pair of objects $A_{n\square}$ and ${}^{\square}A_{n+1}$ of the exact sequence (65). We have two maps, $\overline{\delta}_1$ and $\overline{\delta}_2$, between them, as in (58), of which $\overline{\delta}_1$ is now an isomorphism because of our assumption of exact columns (Corollary 2.1). Let us think of the objects $A_{n\square}$ and ${}^{\square}A_{n+1}$ as the combinations, under the functor $\sum$, of the donor objects, respectively the receptor objects, in the chain of objects and maps in $\mathbf{A}$ lying on one of the "staircases" in (32). Flattened out, such a staircase looks like

$$\dots \xrightarrow{\overline{\delta}_2} {}^{\square}A_{i+1,n-i} \overset{\overline{\delta}_1^{-1}}{\cong} A_{i,n-i\square} \xrightarrow{\overline{\delta}_2} {}^{\square}A_{i,n-i+1} \overset{\overline{\delta}_1^{-1}}{\cong} A_{i-1,n-i+1\square} \xrightarrow{\overline{\delta}_2} \cdots . \tag{66}$$

Here the maps $\overline{\delta}_1, \overline{\delta}_2 : A_{n\square} \to {}^{\square}A_{n+1}$ between our "total" donor and receptor objects arise under $\sum$ from the above maps of (66).

If we compose each arrow $\overline{\delta}_2$ of (66) with the preceding (or the following) isomorphism $\overline{\delta}_1^{-1}$, we may regard (66) as a directed system in $\mathbf{A}$. This suggests that we look at the direct limit $\varinjlim_{i\to-\infty} A_{i,n-i\square} = \varinjlim_{i\to-\infty} {}^{\square}A_{i,n-i+1}$ of that system, if this exists in $\mathbf{A}$.

If in fact $\mathbf{A}$ has countable coproducts (direct sums), then it has general countable colimits (by [3, Proposition 7.6.6], and the fact that, being an abelian category, it has coequalizers) and so, in particular, countable direct limits. Examining how this is proved, we see that in this situation, the direct limit of (66) can be constructed as the cokernel of the map

$$\overline{\delta}_1 - \overline{\delta}_2 : \ \bigoplus_i A_{i,n-i\,\square} \ \longrightarrow \ \bigoplus_i {}^{\square}A_{i,n-i+1}. \tag{67}$$

Now suppose countable coproducts are exact in $\mathbf{A}$, and take $\mathbf{B} = \mathbf{A}$ and $\sum = \bigoplus$. Then (up to a possible change of sign in $\overline{\delta}_1$, which clearly won't change the direct limit of (66)) we see that (67) is just

$$\overline{\delta} : \ A_{n\square} \ \longrightarrow \ {}^{\square}A_{n+1}. \tag{68}$$

In summary: if $\sum = \bigoplus$, then the cokernel of the step (68) in the exact sequence (65) is given by the direct limit of (66).

The *kernel* of (68) does not in general have such a natural description for $\sum = \bigoplus$. But if $\mathbf{A}$ is the category of R-modules (R any ring), that kernel will be zero! Indeed, consider any nonzero $x \in A_{n\square} = \bigoplus_i A_{i,n-i\,\square}$. Let i be the largest integer such that $0 \neq x_i \in A_{i,n-i\,\square}$ (corresponding diagrammatically to the *lowest* position where x has nonzero component on the staircase where it lives). Examining the ${}^{\square}A_{i+1,n-i}$ component of $\overline{\delta}(x)$, we see that this is $\overline{\delta}_1(x_i)$, which is nonzero because $\overline{\delta}_1$ is an isomorphism. So $\overline{\delta}$ is injective, as claimed.

Applying these observations to the exact sequence (65), we get:

6.5. Corollary. *If R is a ring, and $(A_{i,r})$ a double complex of R-modules with exact columns, then its total homology with respect to $\bigoplus$ is described by*

$$A_{n\searrow} \ \cong \ \mathrm{Cok}(\overline{\delta} : A_{n-1\,\square} \to {}^{\square}A_n) \ \cong \ \varinjlim_{i\to-\infty} A_{i,n-i-1\,\square} \ \cong \ \varinjlim_{i\to-\infty} {}^{\square}A_{i,n-i}, \tag{69}$$

where the direct limits are over the system (66). ■

Dualizing the observation following (68), we see that if $\mathbf{A}$ an abelian category with countable direct products and these are exact, and we take $\sum = \prod$, then the kernel of (68) is the *inverse* limit of (66). The *cokernel* is now hard to describe, even for $\mathbf{A}$ the category of R-modules; but I shall develop the description in that case (with $n-1$ in place of n, for convenience) below.

We begin by noting that for general $\mathbf{A}$ and $\sum$, the exactness of (65) at ${}^{\square}A_n$ and $A_{n\searrow}$ tells us that

$$\mathrm{Cok}(\overline{\delta} : A_{n-1\,\square} \to {}^{\square}A_n) \ \cong \ \mathrm{Im}({}^{\square}A_n \to A_{n\searrow}) \ = \ \mathrm{Ker}(A_{n\searrow} \to A_{n\square}). \tag{70}$$

(where the arrows in the last two expressions are intramural maps).

Now let $\mathbf{A}$ again be the category of R-modules, let $(A_{i,r})$ still be a double complex in $\mathbf{A}$ with exact columns, and let its total complex, its total homology, and our related objects be defined using the functor $\sum = \prod$ on $\mathbf{A}$. Suppose an element $x \in A_{n\nwarrow}$ has the property that for every *finite* subset $I \subseteq \mathbb{Z}$, x can be represented by a cycle in A_n which has zero component in $A_{i,n-i}$ for all $i \in I$. Then I will call x a "peekaboo element" (because wherever you look, it isn't there!) The set of these elements forms a submodule of $A_{n\nwarrow}$, which I shall denote $\mathrm{PB}(A_{n\nwarrow})$. I claim that (70), as a submodule of $A_{n\nwarrow}$, is precisely $\mathrm{PB}(A_{n\nwarrow})$.

Let us first show that $\mathrm{PB}(A_{n\nwarrow})$ is contained in (70), looked at as $\mathrm{Ker}(A_{n\nwarrow} \to A_{n\square})$. Suppose $x \in \mathrm{PB}(A_{n\nwarrow})$, and let x be represented by a cycle

$$(x_i) \in \textstyle\prod_i A_{i,n-i}. \tag{71}$$

Since x is a peekaboo element, for any $j \in \mathbb{Z}$ we can modify (x_i) by a boundary $\delta(y_i)$ to get an element having j-th coordinate 0. But doing this changes x_j by $\delta_2(y_j) \pm \delta_1(y_{j-1})$, so if it can send x_j to zero, we must have $x_j \in \delta_2(A_{j,n-j-1}) + \delta_1(A_{j-1,n-j})$, which means that x_j has zero image in $A_{j,n-j\,\square}$. Since we have proved this for arbitrary j, the image of x in $A_{n\square} = \prod A_{i,n-i\,\square}$ is zero, i.e., $x \in \mathrm{Ker}(A_{n\nwarrow} \to A_{n\square})$, as claimed.

Conversely, suppose $x \in A_{n\nwarrow}$ lies in (70), which we now look at as $\mathrm{Im}({}^{\square}A_n \to A_{n\nwarrow})$. This says that x can be represented by a cycle (x_i) as in (71) such that each coordinate x_i is annihilated by both δ_1 and δ_2. We wish to show that for any finite subset $I \subseteq \mathbb{Z}$, our cycle (x_i) can be modified by a boundary so that it becomes zero at each coordinate in I. Suppose inductively that we have been able to achieve zero entries at all coordinates in $I - \{j\}$, where $j = \min(I)$ (corresponding to the *highest* point we are interested in on our staircase), in the process preserving the condition that each coordinate is annihilated by δ_1 and δ_2. For notational simplicity, let us again call this modified element (x_i). In particular, the coordinate x_j is annihilated by δ_1; so by exactness of the columns of our double complex, we can write $x_j = \delta_1(z_{j-1})$ for some $z_{j-1} \in A_{j-1,n-j}$. If we let $(z_i) \in A_{n-1}$ be the element with $j{-}1$-st coordinate z_{j-1} and all other coordinates 0, it is easy to verify that $(x_i) + (-1)^n \delta((z_i))$ has zero coordinates at all indices in I. (The two coordinates that differ from those of (x_i) are the j-th, which has been brought to zero, and the $j{-}1$-st, which we don't care about because $j-1 \notin I$.) To see that all coordinates of this element are still annihilated by δ_1 and δ_2, note that it suffices to prove that $\delta((z_i))$ has this property, for which, by (46) and (51), it will suffice to show that $\delta_2\delta_1(z_{j-1}) = 0$. But $\delta_2\delta_1(z_{j-1}) = \delta_2(x_j) = 0$ by choice of (x_i). This completes the proof that (70) is given by $\mathrm{PB}(A_{n\nwarrow})$.

Inserting into (65) our earlier description of the kernel of $\overline{\delta}$, and this description of its cokernel, we get

6.6. Corollary. *If R is a ring, and $(A_{i,r})$ a double complex of R-modules with exact columns, and we form its total homology objects $A_{n\nwarrow}$ with respect to $\prod$, then for each n we have a short exact sequence*

$$0 \longrightarrow \mathrm{PB}(A_{n\nwarrow}) \longrightarrow A_{n\nwarrow} \longrightarrow (\varprojlim_{i\to\infty} A_{i,n-i\,\square} \cong \varprojlim_{i\to\infty} {}^{\square}A_{i,n-i+1}) \longrightarrow 0, \tag{72}$$

where the inverse limits are over the system (66). ∎

For an example in which the term $\mathrm{PB}(A_n\diagdown)$ of (72) is nonzero, let p be any prime, and consider the double complex of abelian groups,

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} & ,
\end{tikzcd} \tag{73}$$

where the arrows labeled "p" represent multiplication by p, and the vertical equals-signs denote the identity map.

Let n be the common value of the sum of the subscripts on the objects $\mathbb{Z}$ in the lower diagonal string of such objects in (73), so that A_n is the direct product of these objects. Then $A_{n+1} = 0$, so all members of A_n are cycles. For the same reason, the second inverse limit in (72) is zero; so (72) shows that all elements of $A_n\diagdown$ are peekaboo elements. To establish the existence of *nonzero* peekaboo elements, we must therefore show that $A_n\diagdown \neq 0$.

To this end, consider the map σ from A_n to the group $\mathbb{Z}_p$ of *p-adic integers*, given by

$$\sigma((x_i)_{i\in\mathbb{Z}}) \;=\; \textstyle\sum_{i=0}^{\infty} (-1)^{in}\, p^i\, x_i \;\in\; \mathbb{Z}_p. \tag{74}$$

(Note the "cut-off": all x_i with $i<0$ are ignored.)

If $(x_i)\in A_n$ is a boundary, $(x_i)=\delta((y_i))$, then for each i we have $x_i = \delta_2(y_i) + (-1)^{n-1}\delta_1(y_{i-1}) \;=\; p\,y_i \,-\, (-1)^n\, y_{i-1}$. This makes the computation (74) of $\sigma((x_i))$ a "telescoping sum", where all terms cancel except $-(-1)^n y_{-1}$. Thus, for every boundary $(x_i)=\delta((y_i))$, the element $\sigma((x_i))\in\mathbb{Z}_p$ belongs to $\mathbb{Z}$. On the other hand, we can clearly choose elements (hence, cycles) $(x_i)\in A_n$ for which $\sigma((x_i))$ is an arbitrary member of $\mathbb{Z}_p$. Thus, there are cycles in A_n which are not boundaries, hence nonzero peekaboo elements in $A_n\diagdown$.

It is not hard to show, conversely, that every $(x_i)\in A_n$ such that $\sigma(x_i)\in\mathbb{Z}$ is a boundary, and to conclude that $A_n\diagdown \cong \mathbb{Z}_p/\mathbb{Z}$.

(Incidentally, in (73) we could replace $\mathbb{Z}$ by a polynomial ring $k[x]$, for k a field, and the element p by x. In place of the $\mathbb{Z}_p/\mathbb{Z}$ in the final result we would then get $k[[x]]/k[x]$. Regarding this example as a double complex of *k-vector-spaces,* we see that the existence of nonzero peekaboo elements in $A_n\diagdown$ does not require any noncompleteness property of the base ring.)

We now come to our last choice of $\sum$ on the category $\mathbf{A}$ of R-modules, the right truncated product (or "left formal Laurent sum") functor. Thus we let

$$A_n \;=\; (\textstyle\prod_{i<0} A_{i,n-i}) \;\times\; (\bigoplus_{i\geq 0} A_{i,n-i}). \tag{75}$$

(This was example (iii) in the sentence following (47). I use "right" with reference to the subscript i in $\sum X_i = (\prod_{i<0} X_i) \times (\bigoplus_{i\geq 0} X_i)$, thinking of the indices as written in increasing order from left to right. Unfortunately, since in our double complex the first subscript indexes the row, and increases in the downward direction in our diagrams, "*right* truncated" means, for these diagrams, "truncated in the *downward left* direction along each diagonal".) We shall find that this yields the simplest, i.e., most trivial, characterization of the total homology.

Note that if in $\overline{\delta} = \overline{\delta}_2 \pm \overline{\delta}_1 : A_{n-1\,\Box} \to {}^{\Box}A_n,$ we look at the effects of the respective operators $\overline{\delta}_1$ and $\overline{\delta}_2$ on the first subscript of $A_{i,n-1-i\,\Box}$, each operator adds to this subscript a constant, namely 1, respectively 0. We can think of this as saying our operators are each "homogeneous"; but they are homogeneous of distinct degrees, and the operator of higher degree, $\overline{\delta}_1$, is invertible. It follows that $\overline{\delta}$ will be invertible! Indeed, let us write $\overline{\delta} = (1-\varepsilon)(\pm\overline{\delta}_1)$, where $\varepsilon = \pm\overline{\delta}_2\overline{\delta}_1^{\,-1}$, noting that ε is homogeneous of degree -1. Then we see that the formal inverse

$$\overline{\delta}^{\,-1} \;=\; \pm\,\overline{\delta}_1^{\,-1}(1+\varepsilon+\varepsilon^2+\dots) \tag{76}$$

converges on our right truncated product modules (75), and thus gives a genuine inverse to $\overline{\delta}$. Lemma 6.2 now immediately gives:

6.7. Corollary. *Every double complex of R-modules with exact columns has trivial total homology with respect to the right truncated product* (*left formal Laurent sum*) *functor* $\prod_{i<0} \times \bigoplus_{i\geq 0}$. ■

For the *left* truncated product functor $\bigoplus_{i<0} \times \prod_{i\geq 0}$ (the right formal Laurent sum), the kernel and cokernel of δ seem much more difficult to describe in terms of the directed system (66), and I will not try to do so. (Of course, if, reversing Convention 6.4, we take rows rather than columns exact in our double complex of R-modules, the behaviors of left and right truncated products are reversed.)

In a different direction, one finds that for each of the above four constructions $\sum$, on *any* abelian category $\mathbf{A}$ where it is defined and exact, *weakly bounded* double complexes with exact columns always have trivial total homology:

6.8. Corollary. *Let* $\mathbf{A}$ *be any abelian category with countable coproducts, respectively countable products, respectively both, which are exact functors; let* $(A_{i,r})$ *be a weakly bounded double complex in* $\mathbf{A}$ *with exact columns, and suppose we form its total complex with respect to the coproduct functor, respectively, the product functor, respectively, the right- or left-truncated product functor. Then in each case, the induced maps* $\overline{\delta} : A_{n\,\Box} \to {}^{\Box}A_{n+1}$ (*see* (59)) *will be isomorphisms, and hence the total homology will be zero.*

Sketch of Proof The weak boundedness hypothesis has the effect that for each n, the string of objects and maps

$$\dots \xrightarrow{\overline{\delta}_2} {}^{\Box}A_{i+1,n-i} \xleftarrow[\cong]{\overline{\delta}_1} A_{i,n-i\,\Box} \xrightarrow{\overline{\delta}_2} {}^{\Box}A_{i,n-i+1} \xleftarrow[\cong]{\overline{\delta}_1} A_{i-1,n-i+1\,\Box} \xrightarrow{\overline{\delta}_2} \dots \tag{77}$$

breaks up into (generally infinitely many) finite substrings, separated by zero objects. Thus, the map $\overline{\delta} : A_{n\square} \to {}^{\square}A_{n+1}$ becomes a $\sum$-sum of maps from finite direct sums of consecutive donor objects to finite direct sums of consecutive receptor objects. (One verifies this by examining how each of our four functors $\sum$ behaves with respect to decompositions of its domain into finite subfamilies.) On each of these pairs of finite sums, one verifies that the restriction of $\overline{\delta}$ is invertible, by a finite version of the computation (76). A $\sum$-sum of invertible maps is invertible, completing the proof. ■

I have not investigated the behavior of total homology with respect to any other functors $\sum$. In particular, I do not know of any examples of exact functors $\sum : \mathbf{A}^{\mathbb{Z}} \to \mathbf{B}$ satisfying (47) for which the analog of Corollary 6.8 fails. Cf. Corollary 6.3, which says that any such functor does give trivial homology on double complexes having *both* exact rows and exact columns.

7. Further notes

7.1. A FORMALLY SIMPLER APPROACH A more sophisticated formulation of the basic ideas of this note (say, of Definition 1.1 through Lemma 1.7, plus Corollary 4.3 and Lemmas 6.1 and 6.2) would refer to a single object A of an abelian category, possibly graded, with two commuting (or anticommuting) square-zero endomorphisms δ_1 and δ_2. The situation we have been studying would be the particular case where the category is the bigraded additive category of double complexes in our $\mathbf{A}$. (Bigraded because we would allow subscript-shifting as well as subscript-preserving morphisms.) For instance, in the formulation of Corollary 4.3, the vertical exactness assumption would simply take the form $\mathrm{Im}(\delta_1) = \mathrm{Ker}(\delta_1)$, and the diagram obtained, (65), would reduce to what is called in [6] an exact couple:

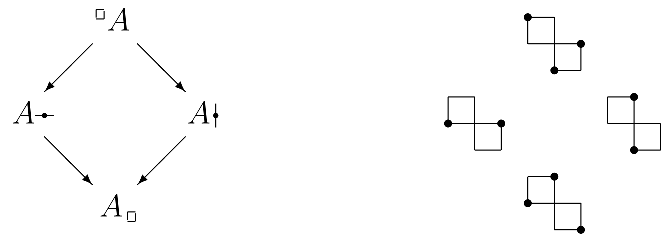
$$\begin{tikzcd} A_{\square} \arrow[rr] & & {}^{\square}A \arrow[dl] \\ & A\diagdown\!\!\!\bullet \arrow[ul] & \end{tikzcd} \; ; \tag{78}$$

alongside which we would have an isomorphism ${}^{\square}A \cong A_{\square}$. But I will not attempt to develop any of the material in this form.

7.2. TRIPLE COMPLEXES At every object A of an ordinary complex, one has a single homology object, while above we have associated to each A in a *double* complex four homology objects, $A_{\square}$, $A\text{-}\!\bullet\!\text{-}$, $A\!\mid\!\!\!\bullet$ and ${}^{\square}A$. What would be the analogous constructions in a triple complex?

To answer this, let us examine how the four constructions we associate to a double complex arise. Let us start with a picture $\begin{smallmatrix}\square & \\ & \square\end{smallmatrix}$, representing an object A of our double complex, together with the three other objects of the complex from which the double complex structure gives us possibly nontrivial maps into A, and the three into which it gives us possibly nontrivial maps from A (cf. (26)); and let us mark with dots those of these objects occurring in the definition of each of our homology objects:

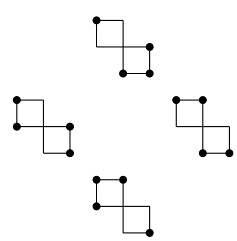
Homology objects Diagrams



(79)

For each of the above diagrams, the associated homology object is the quotient of the intersection of the kernels of the maps from A to the marked objects in the lower square, by the sum of the images in A of the marked objects in the upper square. The dots in the two squares are in each case located so that the maps from the marked objects of the upper square into the marked objects in the lower square via A are all zero, i.e., so that the image in question is indeed contained in the kernel named. On the other hand, the dots in the upper square are in each case as low and as far to the right as they can be without violating this condition, given the positions of the dots in the lower square, and the dots in the lower square are as high and as far to the left as they can be, given the positions of the dots in the upper square.

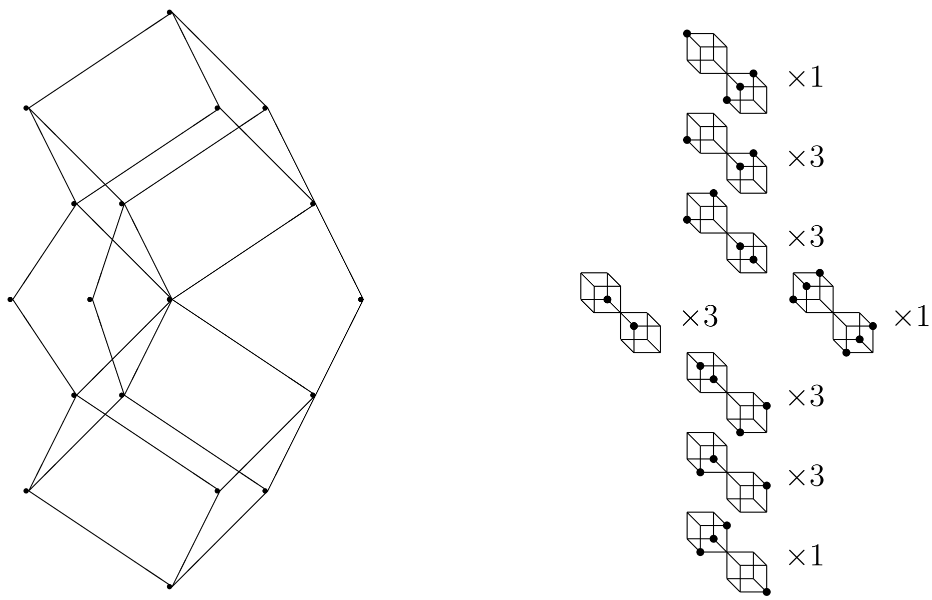
In fact, if we partially order the vertices of our diagrams by considering the arrows of our double complex to go from higher to lower elements, and we supplement our dots in the lower square with all the dots below them under this ordering, and those in the upper square with all the dots above them (noting that so doing does not change the resulting sums of images, or intersections of kernels),



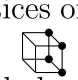
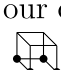
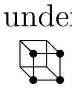
(80)

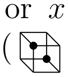
then we see that the arrays of dots in the lower squares of the above four diagrams are precisely the four proper nonempty down-sets (sets closed under $\leq$) of that partially ordered set, and the arrays of dots in the upper squares are (if we momentarily superimpose the upper and lower squares) the *complementary* up-sets.

Knowing this, we can see what the analog should be for triple complexes. One finds that the set of vertices of a cube has 18 proper nonempty down-sets, each with its complementary up-set. Under permutation of the three coordinates, these 18 complementary pairs form 8 equivalence classes. For simplicity, we show on the right below only one representative of each equivalence class, and we again show by dots only maximal elements of each down-set and minimal elements of each up-set, since these correspond to the objects actually needed to compute our homology objects. We mark each diagram $\times 1$ or $\times 3$, to indicate the size of its orbit under permutations of coordinates. On the left, we show the full partially ordered system of these objects. The lines showing the order-relations in that partially ordered set induce intramural maps among our homology objects.



(81)

The reader familiar with lattice theory will recognize the diagram at the left as the free distributive lattice on three generators [5, Figure 19, p. 84]. This is because the distributive lattice of proper nonempty down-sets of the set of vertices of our cube, under union and intersection, is freely generated by the three down-sets ,  and . (These free generators are the three "outer" elements at the middle level of the diagram on the left above, corresponding, in the diagram on the right, to the picture at that level marked $\times 3$. As constructions on our triple complex, they represent the classical homology constructions corresponding to the three axial directions in that complex.)

I have not investigated what extramural maps and exact sequences relate these 18 constructions. I will not propose iconic notations for them like those we have used in studying double complexes; probably the best notation would, rather, involve indexing them by expressions for the elements of a free distributive lattice; e.g., $h_{(x_1 \wedge x_2) \vee x_3}(A_{ijk})$ or $h((x \wedge y) \vee z;\, A_{ijk})$, where $x_1,\ x_2,\ x_3$, or $x,\ y,\ z$, denote the free generators. For informal purposes, though, something like $h($$,\, A_{ijk})$ might occasionally be convenient (as long as we don't go beyond triple complexes).

Are these objects likely to be of use? I don't know!

7.3. Kernel and image ratios J. Lambek [8] (cf. [6, Lemma III.3.1]) associates to any commuting square,

$$\begin{tikzcd} P \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, "f"] & R \arrow[d, "c"] \\ Q \arrow[r, "d"'] & S \end{tikzcd} \qquad \text{where } ca \;=\; f \;=\; db \tag{82}$$

two objects, which he calls the *kernel ratio* and the *image ratio* of the square. To bring out the similarity to the concepts of this paper, let me name them

$$\begin{aligned} P_* \;&=\; \mathrm{Ker}\, f/(\mathrm{Ker}\, a \,+\, \mathrm{Ker}\, b), \\ {}^*S \;&=\; (\mathrm{Im}\, c \,\cap\, \mathrm{Im}\, d)\,/\,\mathrm{Im}\, f\,. \end{aligned} \tag{83}$$

In fact, if the given square is embedded in a double complex which is vertically *and* horizontally exact at P, respectively at S, then we see that $P_* = P_\square$, respectively, ${}^*S = {}^\square S$.

Now suppose we have a commuting diagram with exact rows,

$$\begin{tikzcd} P \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "c"] & T \arrow[d] \\ Q \arrow[r] & S \arrow[r] & U\,. \end{tikzcd} \tag{84}$$

Then we can extend this diagram, by putting in kernels and cokernels of all vertical maps, to a double complex exact in both directions at R and S. Hence Corollary 2.1, applied to c, gives us

$$R_* \;\cong\; R_\square \;\cong\; {}^\square S \;\cong\; {}^*S. \tag{85}$$

The isomorphism $R_* \cong {}^*S$ is proved by Lambek [8] and used to get other results, as I use Corollary 2.1 in §2 above. The constructions $(\ )_*$ and ${}^*(\ )$ have the advantage of being definable with reference to a smaller diagram than my $(\ )_\square$ and ${}^\square(\ )$. They share with these constructions the property of vanishing on any *doubly exact* double complex with finite support. But one doesn't seem to be able to do anything with them without some exactness assumptions. With such assumptions, one gets extramural *iso*morphisms as in Corollary 2.1, but without them, one does not have analogs of the extramural *homo*morphisms of Lemma 1.7.

7.4. Non-abelian groups Lambek proves the results referred to above for not necessarily abelian groups, through he applies them in abelian situations. Note that for non-abelian groups, the exactness of the top row of (84), or something similar, is needed to conclude that ${}^*S$ will be a group, i.e., that the denominator in the definition will be a *normal* subgroup of the numerator. Without that condition, ${}^*S$ is a "homogeneous space”. I likewise noticed when first working out the Salamander Lemma that a version could be stated for not-necessarily-abelian groups, but there were even worse difficulties – e.g., in Definition 1.1, $A_{\square}$ would just be a pointed set, the quotient of the group $\operatorname{Ker} q$ by the left action of the subgroup $\operatorname{Im} c$ and the right action of the subgroup $\operatorname{Im} d$. However, it would certainly be nice to have a tool like the Salamander Lemma for proving noncommutative versions of basic diagram-chasing lemmas, when these hold.

Leicht [11], Kopylov [7], and others have given more general conditions on a category under which Lambek's result holds. In [9], Lambek gets related results for varieties of algebras in the sense of universal algebra satisfying an appropriate Mal'cev-type condition.

8. Two exercises

I have left many key calculations in this note to the reader, including the verification of the Salamander Lemma (Lemma 1.7) itself. My “sketches of proofs” can likewise be regarded “exercises with hints”. Here are two further interesting exercises.

8.1. Building up finite exact double complexes Let $\mathbf{A}$ be an abelian category, and let $\mathbf{A}^{\#}$ denote the abelian category of double complexes in $\mathbf{A}$, where a morphism $f : A \to B$ is a family of morphisms $f_{i,r} : A_{i,r} \to B_{i,r}$ commuting with δ_1 and δ_2. (We are not admitting subscript-shifting morphisms here.) Let $\mathrm{FX} \subseteq \mathrm{Ob}(\mathbf{A}^{\#})$ (standing for “finite exact”) be the class of double complexes with only finitely many nonzero objects, and all rows and columns exact, and $\mathrm{EX} \subseteq \mathrm{FX}$ (“elementary exact”) be the class of double complexes of the form

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd} \tag{86}$$

placed at arbitrary locations in the grid, where the maps among all copies of A are the identity.

(a) Show that FX is the least subclass of $\mathrm{Ob}(\mathbf{A}^{\#})$ containing EX and closed therein under extensions.

Hint: Given an object of FX, prove that one can map it epimorphically to an object of EX as suggested below.

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & {} & 0 &
\end{tikzcd}
\quad \twoheadrightarrow \quad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
. \qquad (87)
$$

To get epimorphicity at A, show that $^{\square}D = 0$.

(b) In contrast, show by example that for a double complex which is *not* assumed to be finite, but nonetheless has exact rows and columns, and has an object D with zero objects immediately below it and to its right, the map of (87) may fail to be an epimorphism.

I expect that an epimorphicity result analogous to (87) should hold for finite exact triple (and higher) complexes. If so, one could get the analog of (a) above, and deduce from this that the 18 constructions of §7.2 (corresponding to the diagrams of (81)) all give zero at objects of a finite triple complex exact in all three directions.

8.2. Complexes with a twist. (a) Suppose we are given a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & & & & \\
{} \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \arrow[dr, bend left] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & F \arrow[r] \arrow[dr, bend right] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & & H \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & {} \\
 & & & & & 0 & 0 & 0 &
\end{tikzcd}
\qquad (88)
$$

in which the columns beginning and ending with 0 are short exact sequences, while the three "rows" (the two ordinary ones, and the one that makes a detour from the top to the bottom) are merely assumed to be complexes.

Obtain a long exact sequence of (mostly horizontal) homology objects

$$\dots \to V_{-\bullet-} \to C_{-\bullet-} \to N_{-\bullet-} \to W_{-\bullet-} \to E_{\dot{|}} \to E_{-\bullet-} \to F_{-\bullet-} \to F_{\dot{|}} \to P_{-\bullet-} \to X_{-\bullet-} \to H_{-\bullet-} \to Q_{-\bullet-} \to \dots\,. \tag{89}$$

(Suggestion: Square off the curved arrows in (88) by inserting the kernel D and the cokernel G of the arrow $E \to F$, and add zeroes above, respectively, below these, getting an ordinary double complex with all columns exact. Apply to this the idea used in deriving Lemma 4.4.)

(b) Consider any double complex (90), in which we have labeled some of the objects to the upper left and lower right of one arrow, $F \to G$:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} & {} & {} & {} & .
\end{tikzcd} \tag{90}$$

Obtain from this a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & \\
{} \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[d] \arrow[dr, bend left] & & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & A \oplus B \oplus D \arrow[r] \arrow[d] & C \oplus E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & L \oplus M \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A \oplus B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & G \arrow[r] \arrow[dr, bend right] & H \oplus K \arrow[r] \arrow[d] & J \oplus L \oplus M \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & & H \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & {} \\
 & & & & 0 & 0 &
\end{tikzcd} \tag{91}$$

taking the arrows to be the sums of "all available" morphisms. (E.g., the arrow $E \to C \oplus E$ is just the inclusion, but the arrow $C \oplus E \to F$ is the sum of the given arrows $C \to F$

and $E \to F$.) Verify that, after a bit of sign-tweaking, (91) satisfies the hypotheses of part (a). Write out, for this diagram, the middle six terms of the exact sequence (89), then one more on either side.

Thus we see that we can extend the 6-term "salamanders" of Lemma 1.7 to longer exact sequences, if we are willing to define more complicated auxiliary objects.

(c) Do these new constructions still have the property of being zero on *bounded exact* double complexes?

9. Acknowledgements, and a final remark

My parents, Lester and Sylvia Bergman, worked together in scientific photography and illustration; and though they never instructed me in the latter, I acquired from them my love of an effective visual display. (But the responsibility for the often inconsistent TEX coding underlying the diagrams of this note is my own.)

Most of this work was done, and a rough draft written, in the Fall of 1972, when I was supported by an Alfred P. Sloan Research Fellowship, and was a guest at the University of Leeds' stimulating Ring Theory Year. In 2007 Anton Geraschenko digitized, and, with my permission, put online, a copy of that draft. His enthusiasm for the material helped spur me to create this better version. I am also indebted to the referee for several thoughtful suggestions, though shortage of time has prevented me from following many of them.

I have not attempted to make this note a "definitive" development of the subject: that should be left to those who actually work with double complexes, and can judge how best to develop the material. Hence the variation, in the present note, between sections where objects are denoted by arbitrary letters and those where they are distinguished by double subscripts (as seemed to give the most readable presentation of one or another topic), the lack of any general notation for intramural and extramural maps, and the choice of vertical arrows that point downward (as in most familiar lemmas proved by diagram chasing) rather than upward (as might be preferable based on systematic considerations).

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