

## LINEAR MAPS ON $k^I$ , AND HOMOMORPHIC IMAGES OF INFINITE DIRECT PRODUCT ALGEBRAS

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ABSTRACT. Let  $k$  be an infinite field,  $I$  an infinite set,  $V$  a  $k$ -vector-space, and  $g : k^I \rightarrow V$  a  $k$ -linear map. It is shown that if  $\dim_k(V)$  is not too large (under various hypotheses on  $\text{card}(k)$  and  $\text{card}(I)$ , if it is finite, respectively countable, respectively  $< \text{card}(k)$ ), then  $\ker(g)$  must contain elements  $(u_i)_{i \in I}$  with all but finitely many components  $u_i$  nonzero.

These results are used to prove that any homomorphism from a direct product  $\prod_I A_i$  of not-necessarily-associative algebras  $A_i$  onto an algebra  $B$ , where  $\dim_k(B)$  is not too large (in the same senses) must factor through the projection of  $\prod_I A_i$  onto the product of finitely many of the  $A_i$ , modulo a map into the subalgebra  $\{b \in B \mid bB = Bb = \{0\}\} \subseteq B$ .

Detailed consequences are noted in the case where the  $A_i$  are Lie algebras.

A version of the above result is proved with the field  $k$  replaced by a commutative valuation ring.

This note resembles [2] in that the two papers obtain similar results on homomorphisms on infinite product algebras; but the methods are different, and the hypotheses under which the methods of this note work are in some ways stronger, in others weaker than those of [2]. Also, in [2] we obtained many consequences from our results, while here we aim for brevity, and after one main result about general algebras, restrict ourselves to a couple of quick consequences for Lie algebras.

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### 1. DEFINITIONS, AND FIRST RESULTS

Let us fix some terminology and notation.

**Definition 1.** *Throughout this note,  $k$  will be a field.*

*By an algebra over  $k$  we shall mean a  $k$ -vector-space  $A$  given with a  $k$ -bilinear multiplication  $A \times A \rightarrow A$ , which we do not assume associative or unital.*

*If  $A$  is an algebra, we define its total annihilator ideal to be*

$$(1) \quad Z(A) = \{x \in A \mid xA = Ax = \{0\}\}.$$

*If  $a = (a_i)_{i \in I}$  is an element of a direct product algebra  $A = \prod_I A_i$ , then we define its support as*

$$(2) \quad \text{supp}(a) = \{i \in I \mid a_i \neq 0\}.$$

*For  $J$  any subset of  $I$ , we shall identify  $\prod_{i \in J} A_i$  with the subalgebra of  $\prod_{i \in I} A_i$  consisting of elements whose support is contained in  $J$ . We also define the subalgebra*

$$(3) \quad A_{\text{fin}} = \{a \in A \mid \text{supp}(a) \text{ is finite}\}.$$

The importance of  $k$ -linear functions on spaces  $k^I$  for the study of homomorphisms on direct product algebras arises from the following curious observation:

**Lemma 2.** *Suppose  $(A_i)_{i \in I}$  is a family of  $k$ -algebras,  $B$  a  $k$ -algebra,  $f : A = \prod_{i \in I} A_i \rightarrow B$  a surjective algebra homomorphism, and  $a = (a_i)_{i \in I}$  an element of  $A$ , and consider the linear map*

$$(4) \quad g : k^I \rightarrow B \quad \text{defined by} \quad g((u_i)) = f((u_i a_i)) \quad \text{for all} \quad (u_i) \in k^I.$$

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Then

- (i) If  $\ker(g)$  contains an element  $u = (u_i)_{i \in I}$  whose support is all of  $I$ , then  $f(a) \in Z(B)$ .
- (ii) More generally, for any  $u \in \ker(g)$ , if we write  $a = a' + a''$ , where  $\text{supp}(a') \subseteq \text{supp}(u)$  and  $\text{supp}(a'') \subseteq I - \text{supp}(u)$ , then  $f(a') \in Z(B)$ .
- (iii) Hence, if  $\ker(g)$  contains an element whose support is cofinite in  $I$ , then  $a$  is the sum of an element  $a' \in f^{-1}(Z(B))$  and an element  $a'' \in A_{\text{fin}}$ .

*Proof.* (i): Given any  $b \in B$ , let us write  $b = f(x)$ , where  $x = (x_i) \in A$ , and compute

$$(5) \quad \begin{aligned} f(a)b &= f(a)f(x) = f(ax) = f((a_i x_i)) = f((u_i a_i u_i^{-1} x_i)) \\ &= f((u_i a_i)) f((u_i^{-1} x_i)) = 0 f((u_i^{-1} x_i)) = 0. \end{aligned}$$

So  $f(a)$  left-annihilates all elements of  $B$ ; and by the same argument with the order of factors reversed, it right-annihilates all elements of  $B$ . Thus,  $f(a) \in Z(B)$ , as claimed.

(ii): Let  $u' \in k^I$  be defined by taking  $u'_i = u_i$  for  $i \in \text{supp}(u)$ , and  $u'_i = 1$  for  $i \notin \text{supp}(u)$ . Thus,  $\text{supp}(u') = I$ ; moreover,  $u'a' = ua$ , whence  $f(u'a') = f(ua) = 0$ . Hence, defining  $g'$  in terms of  $a'$  as  $g$  was defined in (4) in terms of  $a$ , we see that  $\ker(g')$  contains the element  $u'$ , whose support is  $I$ ; so by (i),  $f(a') \in Z(B)$ .

(iii) clearly follows from (ii). □

Motivated by (iii), let us set out to find conditions under which the kernel of a homomorphism on  $k^I$  must contain elements of cofinite support. Here is an easy one.

**Lemma 3.** *Let  $I$  be a set with  $\text{card}(I) \leq \text{card}(k)$ , and  $g : k^I \rightarrow V$  a  $k$ -linear map, for some finite-dimensional  $k$ -vector-space  $V$ . Then there exists  $u \in \ker(g)$  such that  $I - \text{supp}(u)$  is finite.*

*Proof.* By the assumption on  $\text{card}(I)$ , we can choose  $x = (x_i) \in k^I$  whose entries  $x_i$  are distinct. Regarding  $k^I$  as a  $k$ -algebra under componentwise operations, let us map the polynomial algebra  $k[t]$  into it by the homomorphism sending  $t$  to this  $x$ . Composing with  $g : k^I \rightarrow V$ , we get a  $k$ -linear map  $k[t] \rightarrow V$ .

Since  $V$  is finite dimensional, this map has nonzero kernel, so we may choose  $0 \neq p(t) \in k[t]$  such that  $p(x) \in \ker(g)$ . Since the polynomial  $p$  has only finitely many roots,  $p(x_i)$  is zero for only finitely many  $i$ , so  $p(x)$  gives the desired  $u$ . □

Putting together Lemmas 2 and 3, we get

**Proposition 4.** *Let  $k$  be an infinite field, let  $(A_i)_{i \in I}$  be a family of  $k$ -algebras such that the index set  $I$  has cardinality  $\leq \text{card}(k)$ , let  $A = \prod_I A_i$ , and let  $f : A \rightarrow B$  be any surjective algebra homomorphism to a finite-dimensional  $k$ -algebra  $B$ .*

*Then  $B = f(A_{\text{fin}}) + Z(B)$ . (Equivalently,  $A = A_{\text{fin}} + f^{-1}(Z(B))$ .)*

*Hence  $B$  is the sum of  $Z(B)$  and the (mutually annihilating) images of finitely many of the  $A_i$ .*

*Proof.* The first assertion follows immediately from the two preceding lemmas. To get the final assertion, we note that since  $B$  is finite-dimensional, its subalgebra  $f(A_{\text{fin}}) = f(\bigoplus_I A_i) = \sum_I f(A_i)$  must be spanned by the images of finitely many of the  $A_i$ , and since the  $A_i$ , as subalgebras of  $A$ , annihilate one another, so do those images. □

In the next two sections we shall obtain three strengthenings of Lemma 3, two of which weaken the assumption of finite-dimensionality of  $V$ , while the third, instead, weakens the restriction on  $\text{card}(I)$ .

## 2. LARGER-DIMENSIONAL $V$ .

Our first generalization of Lemma 3 will be obtained by replacing the countable-dimensional polynomial ring  $k[t]$  by the rational function field  $k(t)$ , which has dimension  $\text{card}(k)$  over  $k$ . Rational functions are not, strictly speaking, functions; but that will be easy to fudge.

**Lemma 5.** *For each  $c \in k$ , let  $p^{(c)} \in k^k$  be the function which for every  $x \in k - \{c\}$  has  $p^{(c)}(x) = (x - c)^{-1}$ , and at  $c$  has the value 0. Then any nontrivial linear combination of the elements  $p^{(c)}$  has at most finitely many zeroes.*

*Hence if  $I$  is a set of cardinality  $\leq \text{card}(k)$ , and  $g$  is a  $k$ -linear map of  $k^I$  to a  $k$ -vector-space  $V$  of dimension  $< \text{card}(k)$ , then  $\ker(g)$  contains an element  $u$  of cofinite support.*

*Proof.* In  $k(t)$ , any linear combination of elements  $(t - c_1)^{-1}, \dots, (t - c_n)^{-1}$  for distinct  $c_1, \dots, c_n \in k$  ( $n \geq 1$ ), such that each of these elements has nonzero coefficient, gives a *nonzero* rational function

$$(6) \quad a_1(t - c_1)^{-1} + \dots + a_n(t - c_n)^{-1} = g(t)/((t - c_1) \dots (t - c_n)) \quad (\text{where } g(t) \in k[t]).$$

Indeed, to see that (6) is nonzero in  $k(t)$ , multiply by any  $t - c_m$ . Then we can evaluate both sides at  $c_m$ , and we find that the left-hand side then has a unique nonzero term; so we must have  $g(c_m) \neq 0$ . Hence  $g(t)$  is a nonzero element of  $k[t]$ , so (6) is a nonzero element of  $k(t)$ .

If we now take the corresponding linear combination of  $p^{(c_1)}, \dots, p^{(c_n)}$  in  $k^k$ , the result has the value  $g(c)/((c - c_1) \dots (c - c_n))$  at each  $c \neq c_1, \dots, c_n$ . Hence it is nonzero everywhere except at the finitely many zeroes of  $g(t)$ , and some subset of the finite set  $\{c_1, \dots, c_n\}$ .

We get the final assertion by embedding the set  $I$  in  $k$ , so that the  $p^{(c)}$  ( $c \in k$ ) induce elements of  $k^I$ . These will form a  $\text{card}(k)$ -tuple of functions, any nontrivial linear combination of which is a function with only finitely many zeroes. Under a linear map  $g$  from  $k^I$  to a vector space  $V$  of dimension  $< \text{card}(k)$ , some nontrivial linear combination  $u$  of these  $\text{card}(k)$  elements must go to zero, yielding a member of  $\ker(g)$  with the asserted property.  $\square$

(An alternative way to get around the problem that rational functions have poles is to partition  $k$  into two disjoint subsets of equal cardinalities, and use linear combinations of rational functions  $1/(t - c)$  with  $c$  ranging over one of these sets, to get functions on the other.)

For  $k$  countable, we see that the above lemma gives no improvement over the result we got using  $k[t]$ . But in that case, we can improve on Lemma 3 using a different method.

**Lemma 6.** *Suppose the field  $k$  is countably infinite, and  $I$  is a countably infinite set. Then there exists an uncountable-dimensional subspace  $W \subseteq k^I$  such that no nonzero member of  $W$  has infinitely many zeroes.*

*Hence any  $k$ -linear map  $g$  from  $k^I$  to a  $k$ -vector-space  $V$  of at most countable dimension has in its kernel an element  $u$  of cofinite support.*

*Proof.* By Zorn's Lemma, there exists a subspace  $W \subseteq k^I$  maximal for the property that every nonzero member of  $W$  has cofinite support. Let us suppose  $W$  at most countable-dimensional, and use a diagonal argument to obtain a contradiction.

Since  $k$  is countable, if  $W$  is countable-dimensional over  $k$  it is also countable, so we may enumerate its elements,  $w^{(0)}, w^{(1)}, \dots$ ; let us also enumerate the elements of  $I$ :  $i_0, i_1, \dots$ . We can now choose an  $x \in k^I$  which for each natural number  $n$  has value at  $i_n$  different from the values there of the finitely many elements  $w^{(0)}, \dots, w^{(n)}$ .

Clearly  $x \notin W$ . If we look at a nonzero element  $cx + w \in kx + W$ , we see that if  $c \neq 0$ , this can be written  $c(x - w^{(n)})$  for some natural number  $n$ , and will therefore have only finitely many zeroes (some subset of  $\{i_0, \dots, i_{n-1}\}$ ), while if  $c = 0$ ,  $cx + w$  is a nonzero member of  $W$ , and so by assumption has only finitely many zeroes. Thus,  $kx + W$  contradicts the maximality of  $W$ , proving our first assertion.

The final assertion follows as before.  $\square$

*Remarks.* The rational-functions construction of Lemma 5, like the polynomial construction of Lemma 3, gives a vector space  $W \subseteq k^I$  such that for every finite-dimensional subspace of  $W$ , there is a uniform bound on the number of zeroes of its nonzero elements; but our diagonal proof of Lemma 6 does not yield this property. It would be interesting to know whether, for countable  $k$  and  $I$ , there does exist an uncountable-dimensional subspace of  $k^I$  with that property.

Something that one *can* get, by applying our diagonal argument to the  $k$ -algebra structure on  $k^I$ , and not just the  $k$ -vector-space structure, is a subalgebra of  $k^I$  which is a polynomial algebra in uncountably many indeterminates, every nonzero member of which has cofinite support on  $I$ . By the approach of Lemma 5, we can enlarge this to a subalgebra which, modulo the ideal  $(k^I)_{\text{fin}}$ , is a pure transcendental field extension of  $k$  of uncountable transcendence degree. But we don't know of a use for these observations.

### 3. LARGER $I$ .

For our third generalization of Lemma 3, we return to the hypothesis that  $V$  is finite-dimensional, and prove that in that situation, the statement that every linear map  $g : k^I \rightarrow V$  has elements of cofinite support in fact holds for sets  $I$  of much greater cardinality than  $\text{card}(k)$ .

We can no longer get this conclusion by finding an infinite-dimensional subspace  $W \subseteq k^I$  whose nonzero members each have only finitely many zeroes. On the contrary, when  $\text{card}(I) > \text{card}(k)$  (with the latter infinite) there can be no subspace  $W \subseteq k^I$  of dimension  $> 1$  whose nonzero members all have only finitely many zeroes. For if  $(x_i)$  and  $(y_i)$  are linearly independent elements of  $W$ , and we look at the subspaces of  $k^2$  generated by the pairs  $(x_i, y_i)$  as  $i$  runs over  $I$ , then if  $\text{card}(I) > \text{card}(k)$ , at least one of these subspaces must occur at  $\text{card}(I)$  many values of  $i$ , but cannot occur at all  $i$ ; hence some linear combination of  $(x_i)$  and  $(y_i)$  will have  $\text{card}(I)$  zeroes, but not itself be zero. So we must construct our elements of cofinite support in a different way, paying attention to the particular map  $g$ .

Surprisingly, our proof will again use the polynomial trick of Lemma 3; though this time only after considerable preparation. (We could use rational functions in place of these polynomials as in Lemma 5, or the diagonal construction of Lemma 6, but so far as we can see, this would not improve our result, since finite-dimensionality of  $V$  is required by other parts of the proof.)

We remark that the case of Theorem 9 that we will deduce from the result of this section is actually slightly weaker than the corresponding result proved by different methods in [2]. Hence the reader who is only interested in consequences for algebra homomorphisms  $\prod_I A_i \rightarrow B$  may prefer to skip the rather lengthy and intricate argument of this section. On the other hand, insofar as our general technique makes the question, “For what  $k$ ,  $I$  and  $V$  can we say that the kernel of every  $k$ -linear map  $k^I \rightarrow V$  must contain an element of cofinite support?” itself of interest, the result of this section creates an interesting complement to those of §2.

We will assume here familiarity with the definitions of ultrafilter and ultraproduct (given in most books on universal algebra or model theory, and summarized in [2, §14]), and of  $\kappa$ -completeness of an ultrafilter (developed, for example, in [6] or [7], and briefly summarized in the part of [2, §15] preceding Theorem 47).

In the lemma below, we do not yet restrict  $\text{card}(I)$  at all. As a result, we will get functions with zero-sets characterized in terms of finitely many  $\text{card}(k)^+$ -complete ultrafilters, rather than finitely many points. In the corollary to that lemma, we impose a cardinality restriction which forces such ultrafilters to be principal, and so get elements with only finitely many zeroes. The lemma also allows  $k$  to be finite, necessitating a proviso (7) that its cardinality not be too small relative to  $\dim_k(V)$ ; this, too, will go away in the corollary, where, for other reasons, we will have to require  $k$  to be infinite.

In reading the lemma and its proof, the reader might bear in mind that the property (9) makes  $J_0$  “good” for our purposes, while  $J_1, \dots, J_n$  embody the complications that we must overcome. The case of property (9) that we will want in the end is for the element  $0 \in g(k^{J_0})$ ; but in the course of the proof it will be important to consider that condition for arbitrary elements of that set.

**Lemma 7.** *Let  $I$  be a set,  $V$  a finite-dimensional  $k$ -vector space such that*

$$(7) \quad \text{card}(k) \geq \dim_k(V) + 2,$$

*and  $g : k^I \rightarrow V$  a  $k$ -linear map.*

*Then  $I$  may be decomposed into finitely many disjoint subsets,*

$$(8) \quad I = J_0 \cup J_1 \cup \dots \cup J_n$$

*( $n \geq 0$ ), such that*

$$(9) \quad \text{every element of } g(k^{J_0}) \text{ is the image under } g \text{ of an element having support precisely } J_0,$$

*and such that each set  $J_m$  for  $m = 1, \dots, n$  has on it a  $\text{card}(k)^+$ -complete ultrafilter  $\mathcal{U}_m$  such that, letting  $\psi$  denote the factor-map  $V \rightarrow V/g(k^{J_0})$ , the composite  $\psi g : k^I \rightarrow V/g(k^{J_0})$  can be factored*

$$(10) \quad k^I = k^{J_1} \times \dots \times k^{J_n} \rightarrow k^{J_1}/\mathcal{U}_1 \times \dots \times k^{J_n}/\mathcal{U}_n \hookrightarrow V/g(k^{J_0}),$$

*where  $k^{J_m}/\mathcal{U}_m$  denotes the ultrapower of  $k$  with respect to the ultrafilter  $\mathcal{U}_m$ , the first arrow of (10) is the natural projection, and the last arrow is an embedding.*

*Proof.* If  $\text{card}(k) = 2$ , then (7) makes  $V = \{0\}$ , and the lemma is trivially true (with  $J_0 = I$  and  $n = 0$ ); so below we may assume  $\text{card}(k) > 2$ .

There exist subsets  $J_0 \subseteq I$  satisfying (9); for instance, the empty subset. Since  $V$  is finite-dimensional, we may choose a  $J_0$  satisfying (9) such that

$$(11) \quad \text{Among subsets of } I \text{ satisfying (9), } J_0 \text{ maximizes the subspace } g(k^{J_0}) \subseteq V,$$

i.e., such that no subset  $J'_0$  satisfying (9) has  $g(k^{J'_0})$  properly larger than  $g(k^{J_0})$ .

Given this  $J_0$ , we now consider subsets  $J \subseteq I - J_0$  such that

$$(12) \quad g(k^J) \not\subseteq g(k^{J_0}), \text{ and } J \text{ minimizes the subspace } g(k^{J_0}) + g(k^J) \text{ subject to this condition, in the sense that every subset } J' \subseteq J \text{ satisfies either}$$

$$(13) \quad g(k^{J'}) \subseteq g(k^{J_0})$$

or

$$(14) \quad g(k^{J_0}) + g(k^{J'}) = g(k^{J_0}) + g(k^J).$$

It is not hard to see from the finite-dimensionality of  $V$ , and the fact that inclusions of sets  $J$  imply the corresponding inclusions among the subspaces  $g(k^{J_0}) + g(k^J)$ , that such minimizing subsets  $J$  will exist if  $g(k^{J_0}) \neq g(k^I)$ . If, rather,  $g(k^{J_0}) = g(k^I)$ , then the collection of such subsets that we develop in the arguments below will be empty, but that will not be a problem.

Let us, for the next few paragraphs, fix such a  $J$ . Thus, every  $J' \subseteq J$ , satisfies either (13) or (14). However, we claim that there cannot be many *pairwise disjoint* subsets  $J' \subseteq J$  satisfying (14). Precisely, letting

$$(15) \quad e = \dim_k((g(k^{J_0}) + g(k^J))/g(k^{J_0})),$$

we claim that there cannot be  $2e$  such pairwise disjoint subsets.

For suppose we had pairwise disjoint sets  $J'_{\alpha,d} \subseteq J$  ( $\alpha \in \{0,1\}$ ,  $d \in \{1, \dots, e\}$ ) each satisfying (14). Let

$$(16) \quad h_1, \dots, h_e \in g(k^{J_0}) + g(k^J)$$

be a minimal family spanning  $g(k^{J_0}) + g(k^J)$  over  $g(k^{J_0})$ . For each  $\alpha \in \{0,1\}$  and  $d \in \{1, \dots, e\}$ , condition (14) on  $J'_{\alpha,d}$  allows us to choose an element  $x^{(\alpha,d)} \in k^{J'_{\alpha,d}}$  such that

$$(17) \quad g(x^{(\alpha,d)}) \equiv h_d \pmod{g(k^{J_0})}.$$

Some of the  $x^{(\alpha,d)}$  may have support strictly smaller than the corresponding set  $J'_{\alpha,d}$ ; if this happens, let us cure it by replacing  $J'_{\alpha,d}$  by  $\text{supp}(x^{(\alpha,d)})$ : these are still pairwise disjoint subsets of  $J$ , and will still satisfy (14) rather than (13), since after this modification, the subspace  $g(k^{J'_{\alpha,d}})$  still contains  $g(x^{(\alpha,d)}) \notin g(k^{J_0})$ .

We now claim that the set

$$(18) \quad J_0^* = J_0 \cup \bigcup_{\alpha \in \{0,1\}, d \in \{1, \dots, e\}} J'_{\alpha,d}$$

contradicts the maximality condition (11) on  $J_0$ . Clearly  $g(k^{J_0^*}) = g(k^{J_0}) + g(k^J)$  is strictly larger than  $g(k^{J_0})$ . To show that  $J_0^*$  satisfies the analog of (9), consider any  $h \in g(k^{J_0^*}) = g(k^{J_0}) + g(k^J)$ , and let us write it, using the relative spanning set (16), as

$$(19) \quad h = h_0 + c_1 h_1 + \dots + c_e h_e \quad (h_0 \in g(k^{J_0}), c_1, \dots, c_e \in k).$$

Since  $\text{card}(k) > 2$ , we can now choose for each  $d = 1, \dots, e$  an element  $c'_d \in k$  which is neither 0 nor  $c_d$ , and form the element

$$(20) \quad x = (c'_1 x^{(0,1)} + (c_1 - c'_1) x^{(1,1)}) + (c'_2 x^{(0,2)} + (c_2 - c'_2) x^{(1,2)}) + \dots + (c'_e x^{(0,e)} + (c_e - c'_e) x^{(1,e)}).$$

By our choice of  $c'_1, \dots, c'_e$ , none of the coefficients  $c'_d$  or  $c_d - c'_d$  is zero, so  $\text{supp}(x) = \bigcup J'_{\alpha,d}$ . Applying  $g$  to (20), we see from (17) that  $g(x)$  is congruent modulo  $g(k^{J_0})$  to  $c_1 h_1 + \dots + c_e h_e$ , hence, by (19), congruent to  $h$ . By (9), we can find an element  $y$  with support precisely  $J_0$  that makes up the difference, so that  $g(y) + g(x) = h$ . The element  $y + x$  has support exactly  $J_0^*$ ; and since we have obtained an arbitrary  $h \in g(k^{J_0^*})$  as the image under  $g$  of this element, we have shown that  $J_0^*$  satisfies the analog of (9), giving the desired contradiction.

Thus, we have a finite upper bound (namely,  $2e - 1$ ) on the number of pairwise disjoint subsets  $J'$  that  $J$  can contain which satisfy (14). So starting with  $J$ , let us, if it is the union of two disjoint subsets with that property, split one off and rename the other  $J$ , and repeat this process as many times as we can. Then

in finitely many steps, we must get a  $J$  which cannot be further decomposed. Summarizing what we know about this  $J$ , we have

$$(21) \quad g(k^J) \not\subseteq g(k^{J_0}), \text{ every subset } J' \subseteq J \text{ satisfies either } g(k^{J'}) \subseteq g(k^{J_0}) \text{ or } g(k^{J_0}) + g(k^{J'}) = g(k^{J_0}) + g(k^J), \text{ and no two disjoint subsets of } J \text{ satisfy the latter equality.}$$

Let us call any subset  $J \subseteq I - J_0$  satisfying (21) a *nugget*. From the above development, we see that

$$(22) \quad \text{Every subset } J \subseteq I - J_0 \text{ such that } g(k^J) \not\subseteq g(k^{J_0}) \text{ contains a nugget.}$$

The rest of this proof will analyze the properties of an arbitrary nugget  $J$ , and finally show (after a possible adjustment of  $J_0$ ) that  $I - J_0$  can be decomposed into finitely many nuggets  $J_1 \cup \dots \cup J_n$ , and that these will have the properties in the statement of the proposition.

We begin by showing that

$$(23) \quad \text{If } J \text{ is a nugget, then the set } \mathcal{U} = \{J' \subseteq J \mid g(k^{J_0}) + g(k^{J'}) = g(k^{J_0}) + g(k^J)\} \text{ is an ultrafilter on } J.$$

To see this, note that by (21), the complement of  $\mathcal{U}$  within the set of subsets of  $J$  is also the set of complements relative to  $J$  of members of  $\mathcal{U}$ , and is, furthermore, the set of all  $J' \subseteq J$  such that  $g(k^{J'}) \subseteq g(k^{J_0})$ . The latter set is clearly closed under unions and passing to smaller subsets, hence  $\mathcal{U}$ , inversely, is closed under intersections and passing to larger subsets of  $J$ ; i.e.,  $\mathcal{U}$  is a filter. Since  $\emptyset \notin \mathcal{U}$ , while the complement of any subset of  $J$  not in  $\mathcal{U}$  does belong to  $\mathcal{U}$ ,  $\mathcal{U}$  is an ultrafilter.

Let us show next that any nugget  $J$  has properties that come perilously close to making  $J_0 \cup J$  a counterexample to the maximality condition (11) on  $J_0$ . By assumption,  $g(k^{J_0 \cup J})$  is strictly larger than  $g(k^{J_0})$ . Now consider any  $h \in g(k^{J_0 \cup J})$ . We may write

$$(24) \quad h = g(w) + g(x), \quad \text{where } w \in k^{J_0}, x \in k^J.$$

Suppose first that

$$(25) \quad h \notin g(k^{J_0}).$$

From (24) and (25) we see that  $g(x) \notin g(k^{J_0})$ , so  $\text{supp}(x) \in \mathcal{U}$ . Now take any element  $x' \in k^J$  which agrees with  $x$  on  $\text{supp}(x)$ , and has (arbitrary) *nonzero* values on points of  $J - \text{supp}(x)$ . The element by which we have modified  $x$  to get  $x'$  has support in  $J - \text{supp}(x)$ , which is  $\notin \mathcal{U}$  because  $\text{supp}(x) \in \mathcal{U}$ ; hence  $g(x') \equiv g(x) \pmod{g(k^{J_0})}$ , hence by (24),  $g(x') \equiv h \pmod{g(k^{J_0})}$ . Hence by (9), we can find  $z \in k^{J_0}$  with support exactly  $J_0$  such that  $g(z) + g(x') = h$ . Thus,  $z + x'$  is an element with support  $J_0 \cup J$  whose image under  $g$  is  $h$ .

This is just what would be needed to make  $J_0 \cup J$  satisfy (9), if we had proved it for all  $h \in g(k^{J_0 \cup J})$ ; but we have only proved it for  $h$  satisfying (25) (which we needed to argue that  $\text{supp}(x)$  belonged to  $\mathcal{U}$ ).

We now claim that if there were any  $x \in k^J$  with  $\text{supp}(x) \in \mathcal{U}$  satisfying  $g(x) \in g(k^{J_0})$ , then we would be able to complete our argument contradicting (11). For modifying such an  $x$  by any element with complementary support in  $J$ , we would get an element with support exactly  $J$  whose image under  $g$  would still lie in  $g(k^{J_0})$ . Adding to this element the images under  $g$  of all elements of  $k^{J_0}$  with support equal to  $J_0$ , we would get images under  $g$  of certain elements with support exactly  $J_0 \cup J$ . Moreover, since  $J_0$  satisfies (9), these sums would comprise all  $h \in g(k^{J_0})$ , i.e., just those values that were excluded by (25). In view of the resulting contradiction to (11), we have proved

$$(26) \quad \text{If } J \text{ is a nugget, then every } x \in k^J \text{ with } \text{supp}(x) \in \mathcal{U} \text{ satisfies } g(x) \notin g(k^{J_0}).$$

We shall now use the ‘‘polynomial functions’’ trick to show that (26) can only hold if the ultrafilter  $\mathcal{U}$  is  $\text{card}(k)^+$ -complete. If  $k$  is finite,  $\text{card}(k)^+$ -completeness is vacuous, so assume for the remainder of this paragraph that  $k$  is infinite. If  $\mathcal{U}$  is not  $\text{card}(k)^+$ -complete, we can find pairwise disjoint subsets  $J_c \subseteq J$  ( $c \in k$ ) with  $J_c \notin \mathcal{U}$ , whose union is all of  $J$ . Given these subsets, let  $z \in k^J$  be the element having, for each  $c \in k$  and  $i \in J_c$ , the value  $z_i = c$  at  $i$ . Taking powers of  $z$  under componentwise multiplication, we get elements  $1, z, \dots, z^n, \dots \in k^J$ . Since  $V$  is finite-dimensional, some nontrivial linear combination  $p(z)$  of these must be in the kernel of  $g$ . But as a nonzero polynomial,  $p$  has only finitely many roots in  $k$ , say  $c_1, \dots, c_r$ . Thus  $\text{supp}(p(z)) = J - (J_{c_1} \cup \dots \cup J_{c_r})$ . Since  $J \in \mathcal{U}$  and  $J_{c_1}, \dots, J_{c_r} \notin \mathcal{U}$ , we get  $\text{supp}(p(z)) \in \mathcal{U}$ ; but since  $p(z) \in \ker(g)$ , we have  $g(p(z)) = 0 \in g(k^{J_0})$ , contradicting (26). Hence

$$(27) \quad \text{For every nugget } J, \text{ the ultrafilter } \mathcal{U} \text{ of (23) is } \text{card}(k)^+ \text{-complete.}$$

We claim next that (27) implies that for any nugget  $J$ ,

$$(28) \quad \dim_k((g(k^{J_0}) + g(k^J))/g(k^{J_0})) = 1.$$

Indeed, choose any  $x \in k^J$  with support  $J$ , and consider any  $y \in k^J$ . If we classify the elements  $i \in J$  according to the value of  $y_i/x_i \in k$ , this gives  $\text{card}(k)$  sets, so by  $\text{card}(k)^+$ -completeness, one of them, say  $\{i \mid y_i = cx_i\}$  (for some  $c \in k$ ) lies in  $\mathcal{U}$ . Hence  $y - cx$  has support  $\notin \mathcal{U}$ , so  $g(y - cx) \in g(k^{J_0})$ , i.e., modulo  $g(k^{J_0})$ , the element  $g(y)$  is a scalar multiple of  $g(x)$ . So  $g(x)$  indeed spans  $g(k^{J_0}) + g(k^J)$  modulo  $g(k^{J_0})$ .

Let us now choose for each nugget  $J$  an element  $x_J$  with support  $J$ . Thus, by the above observations,  $g(x_J)$  spans  $g(k^{J_0}) + g(k^J)$  modulo  $g(k^{J_0})$ . We claim that

$$(29) \quad \text{For any disjoint nuggets } J_1, \dots, J_n, \text{ the elements } g(x_{J_1}), \dots, g(x_{J_n}) \in V \text{ are linearly independent modulo } g(k^{J_0}).$$

For suppose, by way of contradiction, that we had some relation

$$(30) \quad \sum_{m=1}^n c_m g(x_{J_m}) \in g(k^{J_0}), \quad \text{with not all } c_m \text{ zero.}$$

If  $n > \dim_k(V)$ , then there must be a linear relation in  $V$  among  $\leq \dim_k(V) + 1$  of the  $g(x_{J_m}) \in V$ , so in that situation we may (in working toward our contradiction) replace the set of nuggets assumed to satisfy a relation (30) by a subset also satisfying

$$(31) \quad n \leq \dim_k(V) + 1,$$

and (30) by a relation which they satisfy. Also, by dropping from our list of nuggets in (30) any  $J_m$  such that  $c_m = 0$ , we may assume those coefficients all nonzero.

We now invoke for the third (and last) time the maximality assumption (11), arguing that in the above situation,  $J_0 \cup J_1 \cup \dots \cup J_n$  would be a counterexample to that maximality.

For consider any

$$(32) \quad v \in g(k^{J_0 \cup J_1 \cup \dots \cup J_n}).$$

By (28) and our choice of  $x_{J_1}, \dots, x_{J_n}$ ,  $v$  can be written as the sum of an element of  $g(k^{J_0})$  and an element  $\sum d_m g(x_{J_m})$  with  $d_1, \dots, d_n \in k$ . By (7) and (31),  $\text{card}(k) \geq \dim_k(V) + 2 > n$ , hence we can choose an element  $c \in k$  distinct from each of  $d_1/c_1, \dots, d_n/c_n$  (for the  $c_m$  of (30)), i.e., such that  $d_m - cc_m \neq 0$  for  $m = 1, \dots, n$ . Thus,  $\sum (d_m - cc_m)x_{J_m}$ , which by (30) has the same image in  $V/g(k^{J_0})$  as our given element  $v$ , is a linear combination of  $x_{J_1}, \dots, x_{J_n}$  with *nonzero* coefficients, hence has support exactly  $J_1 \cup \dots \cup J_n$ . As before, we can now use (9) to adjust this by an element with support exactly  $J_0$  so that the image under  $g$  of the resulting element  $x$  is  $v$ . Since  $x$  has support exactly  $J_0 \cup J_1 \cup \dots \cup J_n$ , we have the desired contradiction to (11).

It follows from (29) that there cannot be more than  $\dim_k(V)$  disjoint nuggets; so a maximal family of pairwise disjoint nuggets will be finite. Let  $J_1, \dots, J_n$  be such a maximal family.

In view of (22), the set  $J = I - (J_0 \cup J_1 \cup \dots \cup J_n)$  must satisfy  $g(k^J) \subseteq g(k^{J_0})$ , hence we can enlarge  $J_0$  by adjoining to it that set  $J$ , without changing  $g(k^{J_0})$ , and hence without losing (9). We then have (8).

For  $m = 1, \dots, n$ , let  $\mathcal{U}_m$  be the ultrafilter on  $J_m$  described in (23). To verify the final statement of the proposition, that there exists a factorization (10), note that any element of  $k^I$  can be written  $a^{(0)} + a^{(1)} + \dots + a^{(n)}$  with  $a^{(m)} \in k^{J_m}$  ( $m = 0, \dots, n$ ), hence its image under  $g$  will be congruent modulo  $g(k^{J_0})$  to  $g(a^{(1)}) + \dots + g(a^{(n)})$ . Now the image of each  $g(a^{(m)})$  modulo  $g(k^{J_0})$  is a function only of the equivalence class of  $a^{(m)}$  with respect to the ultrafilter  $\mathcal{U}_m$  (since two elements in the same equivalence class will disagree on a subset of  $J_m$  that is  $\notin \mathcal{U}_m$ , hence their difference is mapped by  $g$  into  $g(k^{J_0})$ ). Hence the value of  $g(a)$  modulo  $g(k^{J_0})$  is determined by the images of  $a$  in the spaces  $k^{J_m}/\mathcal{U}_m$ . This gives the factorization (10). The one-one-ness of the factoring map follows from (29).  $\square$

To get from this a result with a simpler statement, recall that a set  $I$  admits a *nonprincipal*  $\text{card}(k)^+$ -complete ultrafilter only if its cardinality is greater than or equal to a measurable cardinal  $> \text{card}(k)$  [6, Proposition 4.2.7]. (We follow [6] in counting  $\aleph_0$  as a measurable cardinal. Thus, we write ‘‘uncountable measurable cardinal’’ for what many authors, e.g., [7, p.177], simply call a ‘‘measurable cardinal’’.)

Now uncountable measurable cardinals, if they exist, must be enormously large (cf. [7, Chapter 6, Corollary 1.8]). Hence for  $k$  infinite, it is a weak restriction to assume that  $I$  is smaller than all  $\text{card}(k)^+$ -complete cardinals. Under that assumption, the  $\text{card}(k)^+$ -complete ultrafilters  $\mathcal{U}_m$  of Lemma 7 must be principal,

determined by elements  $i_m \in I$ ; so each nugget  $J_m$  contains a minimal nugget  $\{i_m\}$ , and we may use such minimal nuggets in our decomposition (8). The statement of Lemma 7 then simplifies to the next result. (No simplification is possible if  $k$  is finite, since then every ultrafilter is  $\text{card}(k)^+$ -complete, and the only restriction we could put on  $\text{card}(I)$  that would force all  $\text{card}(k)^+$ -complete ultrafilters to be principal would be finiteness; an uninteresting situation. So we exclude the case of finite  $k$ .)

**Corollary 8.** *Let  $k$  be an infinite field,  $I$  a set of cardinality less than every measurable cardinal  $> \text{card}(k)$  (if any exist),  $V$  a finite-dimensional  $k$ -vector space, and  $g : k^I \rightarrow V$  a  $k$ -linear map. Then there exist elements  $i_1, \dots, i_n \in I$  such that, writing  $J_0 = I - \{i_1, \dots, i_n\}$ , we have*

(33) *Every element of  $g(k^{J_0})$  is the image under  $g$  of an element having support precisely  $J_0$ .*

*In particular, applying this to  $0 \in g(k^{J_0})$ ,*

(34) *There exists some  $u = (u_i) \in \ker(g)$  such that  $u_i = 0$  for only finitely many  $i$  (namely  $i_1, \dots, i_n$ ).  $\square$*

Since we have excluded the case where  $k$  is finite, the above corollary did not need condition (7), that  $\text{card}(k) \geq \dim_k(V) + 2$ . We end this section with a quick example showing that Lemma 7 does need that condition.

Let  $k$  be any finite field, and  $I$  a subset of  $k \times k$  consisting of one nonzero element from each of the  $\text{card}(k) + 1$  one-dimensional subspaces of that two-dimensional space (i.e.,  $I$  is a set of representatives of the points of the projective line over  $k$ ). Let  $S \subseteq k^I$  be the two-dimensional subspace consisting of the restrictions to  $I$  of all linear functionals on  $k \times k$ . Since  $k^I$  is  $(\text{card}(k)+1)$ -dimensional,  $S$  can be expressed as the kernel of a linear map from  $k^I$  to a  $(\text{card}(k)-1)$ -dimensional vector space  $V$ .

By choice of  $I$ , every element of  $S$  has a zero somewhere on  $I$ , so  $0 \in g(k^I)$  is not the image under  $g$  of an element having all of  $I$  for support. Hence (9) cannot hold with  $J_0 = I$ . If Lemma 7 were applicable, this would force the existence of a nonzero number of nuggets  $J_m$ . Since  $I$  is finite, the associated ultrafilters would be principal, corresponding to elements  $i_m$  such that all members of  $S = \ker(g)$  were zero at  $i_m$  (by the one-one-ness of the last map of (10)). But this does not happen either: for every  $i \in I$ , there are clearly elements of  $S$  nonzero at  $i$ .

Hence the conclusion of Lemma 7 does not hold for this  $g$ . Note that since  $\dim_k(V) = \text{card}(k) - 1$ , the condition  $\text{card}(k) \geq \dim_k(V) + 2$  fails by just 1.

#### 4. BACK TO HOMOMORPHIC IMAGES OF PRODUCT ALGEBRAS

From the above three results on elements with cofinite support, we can now prove the three cases of

**Theorem 9.** *Assume the field  $k$  is infinite, and let  $(A_i)_{i \in I}$  be a family of  $k$ -algebras,  $B$  a  $k$ -algebra, and  $f : A = \prod_I A_i \rightarrow B$  a surjective  $k$ -algebra homomorphism.*

*Suppose further that either*

- (i)  $\dim_k(B) < \text{card}(k)$ , and  $\text{card}(I) \leq \text{card}(k)$ , or
- (ii)  $\dim_k(B)$ ,  $\text{card}(k)$ , and  $\text{card}(I)$  are all countable, or
- (iii)  $\dim_k(B)$  is finite, and  $\text{card}(I)$  is less than every measurable cardinal  $> \text{card}(k)$ .

*Then*

$$(35) \quad B = f(A_{\text{fin}}) + Z(B).$$

*In fact,  $f$  can be written as the sum  $f_1 + f_0$  of a  $k$ -algebra homomorphism  $f_1 : A \rightarrow B$  that factors through the projection of  $A$  onto the product of finitely many of the  $A_i$ , and a  $k$ -algebra homomorphism  $f_0 : A \rightarrow Z(B)$ .*

*Proof.* We see (35) by combining Lemma 2(iii) with Lemma 5 in case (i), with Lemma 6 in case (ii), and with Corollary 8 in case (iii). The remainder of the proof will be devoted to establishing the final assertion.

In doing so, we shall identify each algebra  $A_{i_0}$  ( $i_0 \in I$ ) with the subalgebra of  $A$  consisting of elements with support in  $\{i_0\}$ . In particular, given  $a = (a_i) \in A$  and  $i_0 \in I$ , the component  $a_{i_0} \in A_{i_0}$  will also be regarded as an element of  $A$ .

As in the proof of Proposition 4,  $f(A_{\text{fin}}) = \sum_I f(A_i)$ ; but since not all of our alternative hypotheses (i)-(iii) have  $B$  finite-dimensional, we need a new argument to show that only finitely many of these summands

are needed in (35). In fact, we shall show that the set

$$(36) \quad I_1 = \{i \in I \mid f(A_i) \not\subseteq Z(B)\}$$

is finite. To this end, let us choose for each  $i \in I_1$  an  $a_i \in A_i$  such that  $f(a_i) \notin Z(B)$ , and let  $a = (a_i)_{i \in I_1} \in \prod_{I_1} A_i \subseteq \prod_I A_i$ . By (35), there exist  $a' \in A_{\text{fin}}$  and  $z \in Z(B)$  such that

$$(37) \quad f(a) = f(a') + z.$$

We claim that  $I_1 \subseteq \text{supp}(a')$ . Indeed, consider any  $i_1 \in I_1$ . Since  $f(a_{i_1}) \notin Z(B)$ , we can find some element of  $B$ , which we write  $f(x)$ , where  $x = (x_i) \in A$ , such that either  $f(x)f(a_{i_1}) \neq 0$  or  $f(a_{i_1})f(x) \neq 0$ . Without loss of generality, assume the latter inequality. Then

$$(38) \quad \begin{aligned} 0 \neq f(a_{i_1})f(x) &= f(a_{i_1}x) = f(a_{i_1}x_{i_1}) = f(ax_{i_1}) \\ &= f(a)f(x_{i_1}) = (f(a') + z)f(x_{i_1}) = f(a'x_{i_1}) = f(a'_{i_1}x_{i_1}). \end{aligned}$$

Hence  $a'_{i_1} \neq 0$ , i.e.,  $i_1 \in \text{supp}(a')$ . So  $I_1$  is contained in  $\text{supp}(a')$ , and so is finite.

Now let  $f_1$ , respectively  $f_0$ , be the maps  $A \rightarrow B$  given by projecting  $A$  to its subalgebra  $\prod_{I_1} A_i$ , respectively  $\prod_{I-I_1} A_i$ , and then applying  $f$ . Thus, these are homomorphisms satisfying  $f = f_1 + f_0$ . Since  $\prod_{I_1} A_i$  and  $\prod_{I-I_1} A_i$  annihilate each other in either order in  $A$ , the same is true of the images of  $f_1$  and  $f_0$  in  $B$ . Now (35), adjusted in the light of (36), says that  $B = f_1(A) + Z(B)$ . Since  $f_0(A)$  annihilates both summands in this expression, it is contained in  $Z(B)$ , as claimed.  $\square$

We remark, in connection with the decomposition  $f = f_1 + f_0$ , that though the sum of two algebra homomorphisms is usually not a homomorphism, it is if the images annihilate one another; in particular, if one of those images is contained in the total annihilator ideal of the codomain. (Cf. [2, Lemma 4].)

For a related result on homomorphic images of *inverse limits* of nilpotent algebras  $A_i$ , see [1].

## 5. APPLICATIONS TO LIE ALGEBRAS

We record in this section some consequences of Theorem 9 for Lie algebras. Note that for  $B$  a Lie algebra, our definition of  $Z(B)$  describes what is called the *center* of  $B$ , and regularly denoted by that symbol.

**Theorem 10** (cf. [2, Theorems 21 and 22]). *Let  $k$  be an infinite field, let  $B$  be a Lie algebra and  $(L_i)_{i \in I}$  a family of Lie algebras over  $k$ , and let  $f : L = \prod_I L_i \rightarrow B$  be a surjective homomorphism of Lie algebras. Suppose also that one of the three conditions (i)-(iii) of Theorem 9 relating  $\text{card}(I)$ ,  $\text{card}(k)$ , and  $\dim_k(B)$  holds.*

*Then if all  $L_i$  are solvable, respectively, nilpotent,  $B$  is as well, and is in fact the sum of its center  $Z(B)$  and the (mutually centralizing) images under  $f$  of finitely many of the  $L_i$ .*

*Proof.* The part of the conclusion after “ $B$  is as well” comes directly from Theorem 9. The preceding part follows because a Lie algebra spanned by finitely many mutually centralizing Lie subalgebras which are solvable or nilpotent (as are both the  $f(L_i)$  and  $Z(B)$ ) is again solvable, respectively, nilpotent.  $\square$

If, instead, of looking at nilpotent or solvable Lie algebras, as in Theorem 10, we assume the  $L_i$  simple, the situation is not as straightforward. Let us call a general (not necessarily Lie) algebra *simple* if it has nonzero multiplication, and has no nonzero proper homomorphic image. A simple algebra  $A$  is necessarily idempotent, i.e., satisfies  $AA = A$  (where by  $AA$  we mean the span of the set of pairwise products of elements of  $A$ ). As noted in [2, Lemma 23], an infinite direct product  $A$  of algebras  $A_i$  which are each idempotent is itself idempotent if and only if there is an integer  $n$  such that in all but finitely many of the  $A_i$ , every element can be written as a sum of  $\leq n$  products. When no such  $n$  exists, so that  $AA$  is a proper ideal of  $A$ , then  $A$  has the nonzero homomorphic image  $A/AA$  with zero multiplication, and this is clearly not a direct product of simple algebras. So for Lie algebras, in the latter situation, a description of the general homomorphic image  $B$  of  $A$  under the conditions of Theorem 9 must combine a direct product of simple algebras with an abelian factor.

But do there exist finite-dimensional simple Lie algebras that require unboundedly many brackets to represent their general element? Probably not. It follows from the result of [5] (or [4, Ch.VIII, §11, Exercise 13(b)]) that in a simple Lie algebra over  $\mathbb{C}$  (or any algebraically closed field of characteristic 0), every element can be written as a single bracket. No example seems to be known of an element of a finite-dimensional simple Lie algebra over any field  $k$  which cannot be so represented, though even for  $k = \mathbb{R}$ ,

the most one knows at present is that every element is a sum of two brackets ([9, Corollary A3.5, p.653], [2, fourth paragraph of §9]).

However, in [2, Theorem 26], using the recent result [3, Theorem A] that every finite-dimensional simple Lie algebra over an algebraically closed field  $k$  of characteristic not 2 or 3 can be *generated* by two elements, we deduce that over any infinite field  $k$  (not necessarily algebraically closed) of characteristic not 2 or 3, every finite-dimensional simple Lie algebra  $L$  contains two elements  $x_1$  and  $x_2$  such that  $L = [x_1, L] + [x_2, L]$ . (For related results, cf. [12].) Combining this fact with Theorem 9 above, we get

**Theorem 11** (cf. [2, Theorem 27]). *Let  $k$  be any infinite field of characteristic not 2 or 3. Let  $B$  be a Lie algebra over  $k$ ,  $(L_i)_{i \in I}$  a family of finite-dimensional simple Lie algebras over  $k$ , and  $f : L = \prod_I L_i \rightarrow B$  a surjective homomorphism of Lie algebras. Suppose, moreover, that one of the three conditions (i)-(iii) of Theorem 9 relating  $\text{card}(I)$ ,  $\text{card}(k)$ , and  $\dim_k(B)$  holds.*

*Then  $f$  factors as  $\prod_I L_i \rightarrow L_{i_1} \times \cdots \times L_{i_n} \cong B$  for some  $i_1, \dots, i_n \in I$ , where the arrow represents the natural projection. Thus,  $B$  is finite-dimensional, and is the direct product of the images under  $f$  of finitely many of the  $L_i$ .*

*Proof.* By [2, Theorem 26], quoted above, every element of each of the  $L_i$  is a sum of at most two brackets, hence the same is true in  $L$ , hence in  $B$ . In particular,  $B$  is idempotent (in Lie algebra language, perfect):  $B = [B, B]$ . Combining this with Theorem 9, we get

$$(39) \quad B = [B, B] = [f(A_{\text{fin}}) + Z(B), f(A_{\text{fin}}) + Z(B)] = [f(A_{\text{fin}}), f(A_{\text{fin}})] \subseteq f(A_{\text{fin}}).$$

Hence  $B = f(A_{\text{fin}})$ , so it is a sum of the mutually annihilating images of finitely many of the simple Lie algebras  $L_i$ ; in particular,  $Z(B) = \{0\}$ . Combining this with the final assertions of Theorem 9 gives the desired conclusions.  $\square$

## 6. ALGEBRAS OVER VALUATION RINGS

Can we extend our results to more general commutative base rings than fields?

If  $R$  is an integral domain with field of fractions  $k$ , and  $f : \prod_I A_i \rightarrow B$  a homomorphism of  $R$ -algebras, and we assume that the  $A_i$  and  $B$  are torsion-free, we might hope that by extending scalars to the field of fractions  $k$  of  $R$ , and applying the preceding results to the extended map, we could get similar conclusions about  $f$ . Unfortunately,  $(\prod_I A_i) \otimes_R k$  is in general much smaller than  $\prod_I (A_i \otimes_R k)$ : the former can be identified with the subalgebra of the latter consisting of those elements whose components admit a common denominator in  $R$ . So though a homomorphism  $\prod_I A_i \rightarrow B$  induces a homomorphism  $(\prod_I A_i) \otimes_R k \rightarrow B \otimes_R k$ , there is no reason to expect this to extend to a homomorphism on  $\prod_I (A_i \otimes_R k)$ , to which we might apply Theorem 9.

If instead we try to generalize the results that go into the proof of Theorem 9, we find it is not hard to extend the proofs of Lemmas 3, 5 and 6 to show that the kernel of a map from  $R^I$  to a free  $R$ -module, or even to a  $k$ -vector-space, of appropriate dimension, contains elements with all but finitely many coordinates *nonzero*. But that is not enough: the obvious analog of Lemma 2(iii) requires  $(u_i)$  to have all but finitely many coordinates *invertible*. Let us set down and prove the analog of that lemma, observing that we can actually make use of a condition intermediate between “all nonzero” and “all invertible”, namely “having a nonzero common multiple”.

**Lemma 12.** *Suppose  $R$  is an integral domain,  $(A_i)_{i \in I}$  a family of  $R$ -algebras,  $B$  an  $R$ -algebra that is torsion-free as an  $R$ -module,  $f : A = \prod_{i \in I} A_i \rightarrow B$  a surjective algebra homomorphism, and  $a = (a_i)_{i \in I}$  an element of  $A$ ; and consider the  $k$ -module homomorphism*

$$(40) \quad g : R^I \rightarrow B \quad \text{defined by} \quad g((u_i)) = f((u_i a_i)) \quad \text{for all} \quad (u_i) \in k^I.$$

*Then*

- (i) *If  $\ker(g)$  contains an element  $u = (u_i)_{i \in I}$  whose entries  $u_i$  have a nonzero common multiple  $r \in R$ , then  $f(a) \in Z(B)$ .*
- (ii) *More generally, given  $u \in \ker(g)$  such that the entries of  $u$  that are nonzero admit a nonzero common multiple  $r \in R$ , if we write  $a = a' + a''$ , where  $\text{supp}(a') \subseteq \text{supp}(u)$  and  $\text{supp}(a'') \subseteq I - \text{supp}(u)$ , then  $f(a') \in Z(B)$ .*
- (iii) *Hence, if  $\ker(g)$  contains an element all but finitely many of whose entries are invertible, then  $a$  is the sum of an element  $a' \in f^{-1}(Z(B))$  and an element  $a'' \in A_{\text{fin}}$ .*

*Proof.* (i) is proved like assertion (i) of Lemma 2, except that where we obtained  $f(a)b = 0$  by a computation involving the coefficients  $u_i^{-1} \in k$ , we now prove  $rf(a)b = 0$  by a computation involving the coefficients  $ru_i^{-1}$  (which lie in  $R$  by choice of  $r$ ). Since  $B$  is torsion-free, the relation  $rf(a)b = 0$  then implies  $f(a)b = 0$ , as required; one gets  $bf(a) = 0$  in the same way. We deduce (ii) from (i) as before.

In (iii) we have returned, for use in an application below, to a hypothesis on invertible elements, so to get this from (ii) we need to see that the condition stated implies that the nonzero entries of  $(u_i)$  have a nonzero common multiple. Such a common multiple is equivalent to a nonzero common multiple of those of the nonzero entries that are not invertible. By assumption, there are only finitely many of these, so they have such a common multiple (e.g., their product).  $\square$

We shall now show that if  $R$  is a commutative valuation ring, we can, under appropriate conditions, use some of our earlier results on linear maps on  $k^I$  to get elements  $u \in R^I$  as in (iii) above. Because of the way we will relate our results to those of the preceding sections, the symbol  $k$  will be used below for the residue field of  $R$ , rather than its field of fractions. The following lemma will be our tool for converting to our present goal certain of our earlier results, namely, those that give *subspaces* of  $k^I$  whose nonzero members have almost all components nonzero.

**Lemma 13.** *Let  $R$  be a commutative valuation ring,  $K$  its field of fractions,  $k$  its residue field,  $I$  a set, and  $\lambda$  a cardinal. Suppose that  $k^I$  has a subspace  $W$  of dimension  $\lambda$ , every nonzero element of which has cofinite support.*

*Then if  $g$  is an  $R$ -linear map from  $R^I$  to a  $K$ -vector-space  $V$  of dimension  $< \lambda$ , there exists an element  $u \in \ker(g)$  all but finitely many of whose entries are invertible in  $R$ .*

*Proof.* Let  $c \mapsto \bar{c}$  be the residue map  $R \rightarrow k$ , and for  $u = (u_i) \in R^I$ , let us similarly write  $\bar{u}$  for  $(\bar{u}_i) \in k^I$ . If we take a basis of the subspace  $W \subseteq k^I$  indexed by  $\lambda$ , and lift its elements to  $R^I$ , we get a  $\lambda$ -tuple of elements  $u^{(\alpha)} = (u_i^{(\alpha)}) \in R^I$  ( $\alpha \in \lambda$ ) whose images  $\bar{u}^{(\alpha)} \in k^I$  have the property that every nontrivial  $k$ -linear combination of them has all but finitely many entries nonzero.

Since  $V$  is  $< \lambda$ -dimensional, some nontrivial  $K$ -linear relation  $\sum_{\alpha \in \lambda} c^{(\alpha)}g(u^{(\alpha)}) = 0$  must hold in  $V$ , where  $c^{(\alpha)} \in K$ , almost all zero. Since  $R$  is a valuation ring, and all but finitely many  $c^{(\alpha)}$  are zero, we can, by multiplying by an appropriate member of  $K$ , assume that all  $c^{(\alpha)}$  lie in  $R$ , but that not all lie in the maximal ideal. Hence  $\sum_{\alpha \in \lambda} c^{(\alpha)}\bar{u}^{(\alpha)} \in k^I$  is a nontrivial linear combination of the  $\bar{u}^{(\alpha)}$ , so as noted in the preceding paragraph, it has all but finitely many entries nonzero. Thus  $u = \sum_{\alpha \in \lambda} c^{(\alpha)}u^{(\alpha)}$  is a member of  $\ker(g)$  with all but finitely many entries invertible.  $\square$

We can now combine the above result with Lemmas 5 and 6. (In contrast, Corollary 8 does not yield a subspace of  $k^I$  of the desired sort, and cannot be modified to do so, for the reasons noted in the second paragraph of §2.)

**Theorem 14.** *Let  $R$  be a commutative valuation ring with infinite residue field  $k$ , and  $f : A = \prod_I A_i \rightarrow B$  a homomorphism from a direct product of  $R$ -algebras to a torsion-free  $R$ -algebra  $B$ . Let us write  $\text{rk}_R(B)$  for the rank of  $B$  as an  $R$ -module; i.e., the common cardinality of all maximal  $R$ -linearly independent subsets of  $B$ .*

*Suppose that  $\text{card}(I) \leq \text{card}(k)$ , and that if  $k$  is uncountable,  $\text{rk}_R(B) < \text{card}(k)$ , while if  $k$  is countably infinite,  $\text{rk}_R(B)$  is countable. Then, as in Theorem 9,*

$$(41) \quad B = f(A_{\text{fin}}) + Z(B),$$

*and  $f$  can be written as the sum of a homomorphism  $f_1$  that factors through the projection to a finite subproduct of  $\prod_I A_i$ , and a homomorphism  $f_0$  with image in  $Z(B)$ .*

*Proof.* Since  $B$  is  $R$ -torsion free, it embeds in the  $K$ -algebra  $B \otimes_R K$ ; so replacing  $B$  with this algebra, we may assume it a  $K$ -algebra. In particular,  $\text{rk}_R(B)$  becomes  $\dim_K(B)$ .

Combining Lemma 13 with Lemma 5 when  $k$  is uncountable, and with Lemma 6 when it is countable, we see that in either case, every  $a \in A$  satisfies the hypothesis of Lemma 12(iii), giving (41)

The final statement about the decomposition of  $f$  is proved exactly as in Theorem 9. (That part of the proof, the final paragraph, does not use the  $k$ -vector-space structure.)  $\square$

It would be interesting to know whether one can get a version of Theorem 14 under a hypothesis similar to that of Theorem 9(iii).

The following example shows that in Theorem 14, one cannot weaken the assumption that  $k$  is infinite to merely say that  $R$  is.

**Lemma 15.** *Let  $R$  be a complete discrete valuation ring with finite residue field (e.g., the ring of  $p$ -adic integers for a prime  $p$ , or the formal power series ring  $k[[t]]$  for  $k$  a finite field), and let  $I$  be an infinite set.*

*Then there exists a surjective  $R$ -algebra homomorphism  $R^I \rightarrow R$  whose kernel contains  $(R^I)_{\text{fin}}$ , and which therefore does not satisfy (41).*

*Sketch of proof.* Let  $p$  be a generator of the maximal ideal of  $R$ . (So in the  $p$ -adic example, “ $p$ ” can be the given prime  $p$ , while in the formal power series example, it can be taken to be  $t$ .)

Then  $R$  is the inverse limit of the system of finite rings  $R/(p^n)$ , hence it admits a structure of compact topological ring. Letting  $\mathcal{U}$  be any nonprincipal ultrafilter on  $I$ , we can take limits of  $I$ -tuples of elements of  $R$  with respect to  $\mathcal{U}$  under this compact topology, and so get a ring homomorphism  $f : R^I \rightarrow R$  defined by  $f(a) = \lim_{\mathcal{U}} a_i \in R$ . (In nonstandard analysis, this is called the “standard part map”; cf. [11, p.82, Theorem 5.1].) Clearly, this homomorphism is surjective; but for  $a \in (R^I)_{\text{fin}}$ , we clearly have  $f(a) = 0$ .  $\square$

Returning to the first paragraph of this section, we remark that there actually do exist commutative rings  $R$  (other than fields) with the property that for certain infinite cardinals  $\mu$ , every  $\mu$ -tuple of nonzero elements of  $R$  has a nonzero common multiple; so that if our index-set  $I$  has cardinality  $\leq \mu$ , we can, as proposed in that paragraph, get a result on homomorphisms from product of algebras  $\prod_I A_i$  to  $R$ -torsion-free  $R$ -algebras  $B$  by tensoring with the field of fractions of  $R$  and applying the results of earlier sections. For example, any nonprincipal ultraproduct of integral domains has common multiples of countable families, and more generally, for any cardinal  $\mu$ , an ultraproduct of integral domains with respect to a  $\mu$ -regular ultrafilter [6, 4.3.2 et seq.] has  $\mu$ -fold common multiples. A different class of examples is given by valuation rings whose value group has cofinality  $\geq \mu$  as an ordered set.

But such exotic rings  $R$  are less often used than general valuation rings.

## 7. CAN THE CARDINALITY ASSUMPTIONS OF §§2-3 BE IMPROVED?

The results of §2 and §3 give sufficient conditions on  $\text{card}(k)$ ,  $\text{card}(I)$  and  $\dim_k(V)$  for the kernel of a map  $g : k^I \rightarrow V$  to have elements with cofinite support. We may ask how close to optimal those results are.

For any  $k$  and any nontrivial  $V$ , if a statement of that sort is to hold, it must require  $\text{card}(I)$  to be less than all measurable cardinals  $\mu > \text{card}(k)$  (if these exist). This is because any  $I$  of cardinality greater than or equal to such a  $\mu$  admits a nonprincipal  $\text{card}(k)^+$ -complete ultrafilter  $\mathcal{U}$ , which makes  $k^I/\mathcal{U}$  one-dimensional (cf. proof of (28) above, or [2, Theorem 49]), and hence embeddable in  $V$ , though the kernel of  $k^I \rightarrow k^I/\mathcal{U}$  consists of elements whose zero-sets lie in  $\mathcal{U}$ , and hence are infinite. Thus, Corollary 8 has the weakest possible hypothesis on  $I$ . However, that corollary is proved under the strong hypothesis that  $\dim_k(V)$  be finite; we don’t know whether the same weak hypothesis on  $I$  can be combined with higher bounds on  $\dim_k(V)$ .

On the other hand, fixing  $k$  and an infinite set  $I$ , and looking at how large  $V$  can be allowed to be, we see that for  $V = k^{\aleph_0}$ , projection of  $k^I$  to a countable subproduct gives a map  $k^I \rightarrow V$  whose kernel has no elements of finite support; so we cannot allow  $\dim_k(V)$  to reach  $\dim_k(k^{\aleph_0})$ . By the Erdős-Kaplansky Theorem [10, Theorem IX.2, p.246], this equals  $\text{card}(k)^{\aleph_0}$ . Now if  $\text{card}(k)$  has the form  $\lambda^{\aleph_0}$  for some  $\lambda$ , then  $\text{card}(k)^{\aleph_0} = \text{card}(k)$ ; so in that case, Lemma 5 gives the weakest possible hypothesis on  $\dim_k(V)$ . Likewise, assuming the Continuum Hypothesis, Lemma 6 is optimal for countable  $k$ . But we don’t know whether for general uncountable  $k$ , or for countably infinite  $k$  in the absence of the Continuum Hypothesis, we can weaken the hypothesis  $\dim_k(V) < \text{card}(k)$ , respectively  $\dim_k(V) \leq \text{card}(k)$ , to  $\dim_k(V) < \text{card}(k)^{\aleph_0}$ ; or to something in between.

Turning to our results on algebras over fields, let us mention that [2, Theorem 19] gives a statement which mixes the weak hypotheses on  $\dim_k(B)$  in parts (i) and (ii) of Theorem 9 of the present note with the weak hypothesis on  $\text{card}(I)$  in part (iii) thereof, but imposes the additional condition that as an algebra,  $B$  satisfy “chain condition on almost direct factors” (defined there). That condition is automatic for finite-dimensional algebras, hence that result subsumes part (iii) of our present theorem, but not parts (i) and (ii). We do not know whether that chain condition can be dropped from the result of [2]; if we could do so, we would have a common generalization of all these results.

Incidentally, most of the results of [2] do not exclude the case where  $\text{card}(I)$  is  $\geq$  a measurable cardinal  $> \text{card}(k)$ , but instead give, in that case, a conclusion in which factorization of  $f : \prod_I A_i \rightarrow B$  through finitely many of the  $A_i$  is replaced by factorization through finitely many ultraproducts of the  $A_i$  with respect to  $\text{card}(k)^+$ -complete ultrafilters. Though similar factorizations for a linear map  $g : k^I \rightarrow B$  appear in Lemma 7 of this note, an apparent obstruction to carrying these over to results on algebra homomorphisms is that our proof of the latter applies the results of §§2-3 not just to a single linear map  $g : k^I \rightarrow B$ , but to one such map for each  $a \in A = \prod_I A_i$ ; and different maps yield different families of ultrafilters. However, one can get around this by choosing finitely many elements  $a_1, \dots, a_d \in A$  whose images under  $f$  span  $B$ , regarding them as together determining a map  $g : k^I \rightarrow B^d$ , applying Lemma 7 to that  $g$ , and then showing that the image under  $f$  of any element in the kernel of all the resulting ultraproduct maps has zero product with the images of  $a_1, \dots, a_d \in A$ , hence lies in  $Z(B)$ . For the sake of brevity we have not set down formally a generalization of Theorem 9(iii) based on this argument.

For other results on cardinality and factorization of maps on products, of a somewhat different flavor, see [8].

## REFERENCES

- [1] George M. Bergman, *Homomorphic images of pro-nilpotent algebras*, preprint, 17pp., October 2009, readable at <http://math.berkeley.edu/~gbergman/papers>.
- [2] ——— and Nazih Nahlus, *Homomorphisms on infinite direct product algebras, especially Lie algebras*, preprint, 32pp., October 2009, readable at <http://math.berkeley.edu/~gbergman/papers>.
- [3] Jean-Marie Bois, *Generators of simple Lie algebras in arbitrary characteristics*, *Mathematische Zeitschrift* **262** (2009) 715–741. <http://arxiv.org/pdf/0708.1711>.
- [4] Nicolas Bourbaki, *Éléments de mathématique*, Fasc. XXXVIII: *Groupes et algèbres de Lie*, chapitres VII–VIII. Translated by Andrew Pressley in *Lie Groups and Lie Algebras. Chapters 7–9*. MR **56#12077**, MR **2005h:17001**.
- [5] Gordon Brown, *On commutators in a simple Lie algebra*, *Proc. Amer. Math. Soc.* **14** (1963) 763–767. MR **27#3676**.
- [6] C. C. Chang and H. J. Keisler, *Model Theory*, 3rd ed., *Studies in Logic and the Foundations of Mathematics*, v.73. North-Holland, 1990. MR **91c:03026**.
- [7] F. R. Drake, *Set Theory: An Introduction to Large Cardinals*, *Studies in Logic and the Foundations of Mathematics*, v.76, Elsevier, 1974. Zbl **0294.02034**.
- [8] Andrzej Ehrenfeucht, Siemion Fajtlowicz, and Jan Mycielski, *Homomorphisms of direct powers of algebras*, *Fund. Math.* **103** (1979) 189–203. MR **81b:08002**.
- [9] Karl H. Hofmann and Sidney A. Morris, *The Lie Theory of Connected Pro-Lie Groups. A structure theory for pro-Lie algebras, pro-Lie groups, and connected locally compact groups*, EMS Tracts in Mathematics, 2. European Mathematical Society, 2007. MR **2008h:22001**.
- [10] Nathan Jacobson, *Lectures in Abstract Algebra. Volume II: Linear algebra*, Van Nostrand, 1953, and Springer GTM, No. 31, 1975. MR **14,837e** and MR **51#5614**.
- [11] Peter A. Loeb, *Nonstandard analysis and topology*, pp.77-86 in *Nonstandard Analysis (Edinburgh, 1996)*, ed. Leif Arkeryd, Nigel Cutland, and C. W. Henson. Kluwer, 1997. MR **1603230**.
- [12] Nazih Nahlus, *On  $L = [L, a] + [L, b]$  and  $x = [a, b]$  in split simple Lie algebras*, to be written, title tentative.

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