Comments, corrections, and related references welcomed!

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HOMOMORPHISMS ON INFINITE DIRECT PRODUCT ALGEBRAS, ESPECIALLY LIE ALGEBRAS

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Abstract. We study surjective homomorphisms $f : \prod_I A_i \to B$ of not-necessarily-associative algebras over a commutative ring k, for I a generally infinite set; especially when k is a field and B is countable-dimensional over k.

Our results have the following consequences when k is an infinite field, the algebras are Lie algebras, and B is finite-dimensional:

If all the Lie algebras A_i are solvable, then so is B.

If all the Lie algebras A_i are nilpotent, then so is B.

If k is not of characteristic 2 or 3, and all the Lie algebras A_i are finite-dimensional and are direct products of simple algebras, then (i) so is B, (ii) f splits, and (iii) under a weak cardinality bound on I, f is continuous in the pro-discrete topology. A key fact used in getting (i)-(iii) is that over any such field, every finite-dimensional simple Lie algebra L can be written $L = [x_1, L] + [x_2, L]$ for some $x_1, x_2 \in L$, which we prove from a recent result of J. M. Bois.

The general technique of the paper involves studying conditions under which a homomorphism on $\prod_I A_i$ must factor through the direct product of finitely many ultraproducts of the A_i.

Several examples are given, and open questions noted.

1. Introduction.

In this note, an *algebra* over a commutative associative unital ring k means a k-module A given with a k-bilinear multiplication $A \times A \to A$, which we do not assume associative or unital. We shall assume k fixed, and "algebra" will mean "k-algebra" unless another base ring is specified. "Countable" will be used in the broad sense, which includes "finite". "Direct product" will be used in the sense sometimes called "complete direct product".

Let us sketch our method of approach in a somewhat simpler case than we will eventually be considering. It is easy to show that if an algebra B is not a nontrivial direct product, and has no nonzero elements annihilating all of B, then any surjective homomorphism $f : A_1 \times A_2 \to B$ from a direct product of two algebras onto B must factor through the projection onto one of A_1 or A_2. It follows that for such B, a homomorphism $f : \prod_I A_i \to B$ from an arbitrary direct product onto B will factor through an ultraproduct $\prod_I A_i / \mathcal{U}$.

For this to be useful, we need to know something about such ultraproducts. Assume k a field. There are three main cases:

First, $\mathcal{U}$ may be a principal ultrafilter. Then $\prod_I A_i / \mathcal{U}$ can be identified with one of the A_i, and f factors through the projection to that algebra.

Second, $\mathcal{U}$ may be a nonprincipal ultrafilter that is not $\mathrm{card}(k)^+$-complete. Then $\prod_I A_i / \mathcal{U}$ is an algebra over the ultrapower $K = k^I / \mathcal{U}$, and for such $\mathcal{U}$, K is an uncountable-dimensional extension field of k (Theorem 47). We shall see (Proposition 9) that if we map a K-algebra A onto a k-algebra B having nonzero multiplication, uncountable dimensionality of K forces B to be uncountable-dimensional over k as well. Hence, if we restrict attention to maps onto countable-dimensional B, this case does not occur.

Finally, $\mathcal{U}$ may be a nonprincipal but $\mathrm{card}(k)^+$-complete ultrafilter. If k is finite, $\mathrm{card}(k)^+$-completeness is vacuous (implicit in the definition of an ultrafilter), and we cannot prove much in that case; we mainly

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note open questions. If k is infinite, on the other hand, the $\mathrm{card}(k)^+$-complete case "almost" does not occur: For such an ultrafilter to exist, the index set I must be of cardinality at least an uncountable measurable cardinal, and it is known that if such cardinals exist, they must be extremely large ("inaccessible", and more), and that the nonexistence of such cardinals is consistent with ZFC, the standard axiom system of set theory.

Nevertheless, if $\mathcal{U}$ *is* such an ultrafilter, the behavior of $\prod_I A_i/\mathcal{U}$ is almost as good as when $\mathcal{U}$ is principal. This case can be subdivided in two: If the dimensions of the A_i as k-algebras are not themselves extremely large cardinals, then that ultraproduct will again be isomorphic (though not by a projection) to one of the A_i (Theorem 48), and so will inherit all properties assumed for these. Without any restriction on the dimensions of the A_i, the ultraproduct will still satisfy many important properties that hold on the A_i, e.g., simplicity, nilpotence, or (in the Lie case) solvability (Propositions 49 and 50), again allowing us to get strong conclusions about the image B of f.

Fortunately, the proofs of our main results do not require separate consideration of all these cases, but mainly the distinction between the $\mathrm{card}(k)^+$-complete case (which includes the principal case), and the non-$\mathrm{card}(k)^+$-complete case (which, as indicated, we rule out).

The above sketch assumed that B was not a nontrivial direct product and had no nonzero elements annihilating all of B. We use these hypotheses in the early sections of the paper, but introduce in §6 a weaker hypothesis on B ("chain condition on almost direct factors") yielding more general statements.

The results proved below about Lie algebras were conjectured several years ago by the second author (under the assumption that the A_i were finite-dimensional, and the base field k algebraically closed of characteristic 0).

We obtain some similar results, but by different methods, in [6].

Standard definitions and facts about ultrafilters, ultraproducts, and ultrapowers, assumed from §3 on, are reviewed in an appendix, §14. The more exotic topics of κ-complete ultrafilters and uncountable measurable cardinals, and some results on these, are presented in another appendix, §15, and used from §5 on.

In §11, we give some tangential results on concepts introduced in the preceding sections. A number of examples showing ways in which our results cannot be strengthened are collected in §12, and in §13, we note some open questions and directions for further investigation.

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2. Some preliminaries.

Our statements in the above sketch on when a map $f: A_1 \times A_2 \to B$ must factor through A_1 or A_2, on the one hand, and on when a surjective homomorphism $A \to B$ of k-algebras such that A admits a structure of K-algebra for a "large" extension field K of k forces B to be large, on the other, both had hypotheses restricting the amount of "zero multiplication" in the structure of B. To see why such limitations are needed, note that if k is a field, and the A_i are k-algebras whose multiplications are the zero map, and B is, say, the one-dimensional zero-multiplication k-algebra, then homomorphisms $\prod_I A_i \to B$ are arbitrary linear functionals on $\prod_I A_i$, and these need not satisfy the conclusions of either statement. Homomorphisms based on zero multiplication are, from our point of view, very unruly, and we shall "work around" that phenomenon, in various ways, throughout this paper.

To refer conveniently to that phenomenon, let us make

Definition 1. *If A is an algebra, we define its* total annihilator ideal *to be*

(1) $\quad Z(A) \;=\; \{x \in A \mid xA = Ax = \{0\}\}.$

When A is a Lie algebra, $Z(A)$ is thus the *center* of A, and our notation agrees with standard notation for the center. (But when A is an associative algebra, this notation conflicts with the common notation for the center of A.)

Let us also make explicit that the definition of simple algebra excludes zero multiplication:

Definition 2. *An algebra A will be called* simple *if it is nonzero, has nonzero multiplication, and has no proper nonzero homomorphic images.*

A simple algebra A must be idempotent, $AA = A$, and have zero total annihilator, $Z(A) = \{0\}$, since AA and $Z(A)$ are always ideals.

By AA, above, we of course mean the set of sums of products of pairs of elements of A. More generally,

Definition 3. *For any k-submodules A', A'' of an algebra A, we will denote by $A'A''$ the k-submodule of A consisting of all sums of products $a'a''$ $(a' \in A',\ a'' \in A'')$.*

Below, we will always write AA rather than A^2, to avoid confusion with $A \times A$.

The next lemma shows that the total annihilator ideal leads to a way that algebra homomorphisms can be "perturbed", which we will have to take account of in many of our results. (In this lemma, we explicitly write "k-algebra homomorphism" because a k-module homomorphism is also mentioned. Elsewhere, "homomorphism" will be understood to mean k-algebra homomorphism unless the contrary is stated.)

Lemma 4. *Let A and B be k-algebras, $f : A \to B$ a k-algebra homomorphism, and $h : A \to Z(B)$ a k-module homomorphism. Then the following conditions are equivalent:*

(i) *$f + h$ is a k-algebra homomorphism.*

(ii) $AA \subseteq \ker(h)$.

(iii) *h is a k-algebra homomorphism.*

Proof. We will show (i) $\iff$ (ii). The equivalence (iii) $\iff$ (ii) will follow as the $f = 0$ case of that result.

Since $f + h$ is k-linear, (i) will hold if and only if $f + h$ respects multiplication, i.e., if and only if for all $a, a' \in A$ we have $f(aa') + h(aa') = (f(a) + h(a))(f(a') + h(a'))$. Subtracting from this the equation $f(aa') = f(a)f(a')$, and noting that all the terms remaining on the right are two-fold products in which at least one factor is a value of h, and hence lies in $Z(B)$, making those products 0, we see that (i) is equivalent to $h(aa') = 0$, i.e., (ii). □

Here, in somewhat sharpened form, is the fact stated in the second paragraph of §1.

Lemma 5. *Let B be a nonzero algebra. Then the following conditions are equivalent:*

(i) *Every surjective homomorphism $f : A_1 \times A_2 \to B$ from a direct product of two algebras onto B factors through the projection of $A_1 \times A_2$ onto A_1, or through the projection onto A_2.*

(ii) *B is not a sum $B_1 + B_2$ of two nonzero mutually annihilating subalgebras, i.e., nonzero subalgebras B_1, B_2 such that $B_1B_2 = B_2B_1 = \{0\}$.*

(iii) *$Z(B) = \{0\}$, and B is not a direct product of two nonzero subalgebras.*

In particular, these conditions hold when B is a simple algebra.

Proof. We shall show that $\neg$(i) $\iff$ $\neg$(ii) $\iff$ $\neg$(iii).

If a surjective homomorphism $f : A_1 \times A_2 \to B$ does not factor through either projection map, then $B_1 = f(A_1)$ and $B_2 = f(A_2)$ are both nonzero, and so give a counterexample to (ii). Conversely, given B_1 and B_2 as in $\neg$(ii), the map $B_1 \times B_2 \to B$ given by the sum of the inclusions will be an algebra homomorphism that establishes $\neg$(i).

For B_1 and B_2 as in $\neg$(ii) we get $\neg$(iii) by noting that if they have nonzero intersection, that intersection is a nonzero submodule of $Z(B)$, while if they have zero intersection, then $B \cong B_1 \times B_2$ as algebras.

Finally, in the situation of $\neg$(iii), if $Z(B) \neq \{0\}$ then the equation $B = B + Z(B)$ yields $\neg$(ii), while if B is a direct product of nonzero subalgebras, then those subalgebras give the required B_1 and B_2.

The final assertion holds because a simple algebra satisfies (iii). □

The next lemma strengthens the above result a bit, so as to give interesting information in the case where $Z(B) \neq \{0\}$ as well. We shall not use the result in this form, but it will eventually motivate the transition to the approach of §6.

We remark that the implication (iii$'$) $\Longrightarrow$ (ii$'$) below is not a trivial consequence of (iii) $\Longrightarrow$ (ii) above, because dividing an algebra B by its total annihilator ideal $Z(B)$ does not in general produce an algebra with zero total annihilator ideal, to which we could apply the latter result.

Lemma 6. *Let B be a nonzero algebra. Then the conditions* (i$'$) *and* (ii$'$) *below are equivalent.*

(i$'$) *If a homomorphism $f : A_1 \times A_2 \to B$, when composed with the natural map $B \to B/Z(B)$, gives a surjection, then that composite map factors through the projection of $A_1 \times A_2$ onto A_1, or onto A_2.*

(ii$'$) *B is not equal to the sum $B_1 + B_2$ of two mutually annihilating subalgebras neither of which is contained in $Z(B)$.*

Moreover, the above conditions are implied *by*
(iii′) $B/Z(B)$ *is not a direct product of two nonzero subalgebras.*

Sketch of proof. The surjectivity condition in the hypothesis of (i′) says that $f(A_1) + f(A_2) + Z(B) = B$. With this in mind, we can get the equivalence of (i′) and (ii′) as in the preceding result: given a counterexample to (i′), the subalgebras $B_1 = f(A_1) + Z(B)$ and $B_2 = f(A_2) + Z(B)$ give a counterexample to (ii′), while given a counterexample to (ii′), the induced map $B_1 \times B_2 \to B$ is a counterexample to (i′).

We complete the proof by showing that $\neg$(ii′) $\implies$ $\neg$(iii′). Given B_1 and B_2 contradicting (ii′), enlarge them to $B_1 + Z(B)$ and $B_2 + Z(B)$ if necessary, so that they each contain $Z(B)$; this clearly preserves the conditions assumed. It is now easy to show that their images in $B/Z(B)$ contradict (iii′): Those images are subalgebras which sum to the whole algebra and annihilate one another on both sides, so it suffices to show that they have zero intersection. Since B_1 and B_2 both contain $Z(B)$, an element of the intersection of their images in $B/Z(B)$ will arise from an element $x \in B_1 \cap B_2$. But such an element will annihilate both B_2 and B_1, hence will annihilate their sum, B, i.e., it will lie in $Z(B)$, so the element of $B/Z(B)$ that it yields is indeed zero. □

The above implication (iii′) $\implies$ (ii′) is not reversible. For example, let B be the associative or Lie algebra spanned by the matrix units e_{12}, e_{13}, e_{23} within the algebra $M_3(k)$ of 3×3 matrices over a field k. (As a Lie algebra, B is sometimes called the *Heisenberg algebra*.) We find that $Z(B) = k\, e_{13}$, and that $B/Z(B)$ is a 2-dimensional k-vector space with zero multiplication, hence is the direct product of any two one-dimensional subspaces; so (iii′) fails. But we claim that (ii′) holds. Indeed, if $B = B_1 + B_2$, and neither summand is contained in $Z(B)$, then we can find $b_1 \in B_1$ and $b_2 \in B_2$ which are linearly independent modulo $Z(B)$. It is then not hard to show that in $[b_1, b_2] = b_1 b_2 - b_2 b_1$, the coefficient of e_{13} is given by a determinant of the coefficients of e_{12} and e_{23} in those two elements, and hence is nonzero; so B_1 and B_2 do not annihilate one another, either as Lie or as associative algebras.

By Lemma 6, condition (i′) also holds for this example. Now a homomorphism from a direct product algebra $A_1 \times A_2$ onto the zero-multiplication algebra $B/Z(B)$ can fail to factor through the projection onto A_1 or A_2 (e.g., when the latter are each the one-dimensional zero-multiplication algebra). Yet condition (i′) shows that *if* such a homomorphism arises as a composite $A_1 \times A_2 \to B \to B/Z(B)$, it must so factor.

What this example shows us is that though $Z(B)$ is "trivial", in that its elements have zero multiplication with everything, it cannot be ignored in studying the multiplicative structure of B and the properties of homomorphisms onto B, because elements outside it can have nonzero product lying in it.

3. Factoring homomorphisms through ultraproducts.

Suppose an algebra B satisfies the equivalent conditions of Lemma 5, and we map an infinite direct product $\prod_I A_i$ onto B. Then, since for every subset $J \subseteq I$ we have $\prod_I A_i \cong (\prod_J A_i) \times (\prod_{I-J} A_i)$, Lemma 5 gives us a vast family of factorizations of our homomorphism. How these fit together is described (in a general set-theoretic setting) in the next lemma. In stating it, we assume acquaintance with the concepts of filter, ultrafilter, reduced product and ultraproduct, summarized in §14.

Lemma 7. *Suppose* $(A_i)_{i \in I}$ *is a family of nonempty sets,* B *is a set, and* $f : A = \prod_I A_i \to B$ *is a set map, whose image has more than one element. Then the following conditions are equivalent:*

(a) *For every subset* $J \subseteq I$, *the map* f *factors either through the projection* $A \to \prod_{i \in J} A_i$, *or through the projection* $A \to \prod_{i \in I-J} A_i$.

(b) *The map* f *factors through the natural map* $A \to A/\mathcal{U}$, *where* $\mathcal{U}$ *is an ultrafilter on the index set* I, *and* $A/\mathcal{U} = \prod_I A_i/\mathcal{U}$ *denotes the ultraproduct of the* A_i *with respect to this ultrafilter.*

When this holds, the ultrafilter $\mathcal{U}$ *is uniquely determined by* f.

Proof. Not yet assuming either (a) or (b), but only the initial hypothesis, let

(2) $$\mathcal{F} \;=\; \{J \subseteq I \mid\ f \text{ factors through the projection } A \to \textstyle\prod_{i \in J} A_i\}.$$

Thus, a subset $J \subseteq I$ belongs to $\mathcal{F}$ if and only if for all $a = (a_i) \in A$, $f(a)$ is unchanged on making arbitrary changes in the coordinates of a indexed by the elements of the *complementary* set $I - J$. Now if the value of $f(a)$ is unchanged on changing coordinates lying in a given subset, it is unchanged on changing coordinates in any smaller subset; and if is unchanged on changing coordinates in each of two subsets, then it is unchanged on changing coordinates in the union of those two sets. Translating these observations into statements about the family $\mathcal{F}$ of complements of sets with that property, we see that $\mathcal{F}$ is closed under intersections and enlargement, i.e., $\mathcal{F}$ is a filter on I.

Looking at the definition of the reduced product of a family of sets with respect to a filter on the index set, we see that $\mathcal{F}$ is the largest filter such that f factors through the natural map of A to the reduced product $A/\mathcal{F}$.

The fact that the image of f has more than one element shows that the value of $f(a)$ is not unchanged under arbitrary modification of all coordinates of a; so $\mathcal{F}$ does not contain the empty set, i.e., it is a proper filter.

Finally, we note that condition (a) is equivalent to saying that for each $J \subseteq I$, either J or its complement lies in $\mathcal{F}$; i.e., that $\mathcal{F}$ is an ultrafilter, which we rename $\mathcal{U}$. The equivalence of (a) and (b), and the final assertion, now follow. □

Combining the above with Lemma 5, we get

Proposition 8. *The equivalent conditions* (i)-(iii) *of Lemma 5 on a nonzero algebra B are also equivalent to:*

(iv) *Every surjective homomorphism $f : \prod_I A_i \to B$ from an arbitrary direct product of algebras to B factors through the natural map of that product onto the ultraproduct $\prod_I A_i/\mathcal{U}$, for some ultrafilter $\mathcal{U}$ on I.*

When this holds, the ultrafilter $\mathcal{U}$ is uniquely determined by f.

Proof. Since (i) is the $I = \{1, 2\}$ case of (iv), we have (iv) $\Longrightarrow$ (i).

Conversely, if B satisfies (i), and we have a homomorphism $f : A = \prod_I A_i \to B$, then for every $J \subseteq I$ we can apply (i) to the decomposition $A = (\prod_J A_i) \times (\prod_{I-J} A_i)$, and conclude that f factors through the projection of A to one of these subproducts. Lemma 7 now yields (iv), and the final assertion. □

4. Extending algebra structures.

We now come to the other tool referred to in §1.

Proposition 9. *Suppose $\varphi : k \to K$ is a homomorphism of commutative rings, A is a K-algebra, B is a k-algebra, and $f : A \to B$ is a surjective homomorphism as k-algebras* (*under the k-algebra structure on A induced by its K-algebra structure*).

Then the kernel of the composite map $A \to B \to B/Z(B)$ is an ideal of A, not only as a k-algebra, but as a K-algebra. Hence $B/Z(B)$ acquires a K-algebra structure (*unique for the property of making that composite map a homomorphism of K-algebras*).

Proof. It will suffice to show that for $a \in A$ and $c \in K$, if $f(a) \in Z(B)$, then $f(c\,a) \in Z(B)$. So we must show that $f(c\,a)$ annihilates on both sides an arbitrary element of B, which by surjectivity of f we can write $f(a')$ $(a' \in A)$. To do this, we compute: $f(c\,a)f(a') = f(c\,a\,a') = f(a)f(c\,a') = 0$, the last step by the assumption that $f(a) \in Z(B)$. The same calculation works for the product in the opposite order, completing the proof. □

Remark: If A were unital, then any *ring*-theoretic ideal of A, being closed under multiplication by $K \cdot 1$, would be a K-algebra ideal. In that situation, moreover, $Z(B)$ would be trivial, since no nonzero element of B could be annihilated by $f(1)$ on either side. The above result shows that somehow, lacking unitality of A, we can make up for it at the other end by dividing out by $Z(B)$.

5. First results on homomorphic images of infinite direct product algebras.

In this section we will use the above tools to obtain a couple of results on homomorphic images of direct products of Lie and other algebras, under the assumption that $\mathrm{card}(I)$ is less than any measurable cardinal $> \mathrm{card}(k)$. (That condition holds vacuously, of course, if no such measurable cardinals exist.) We assume from here on the material of the final appendix, §15, on measurable cardinals, and κ^+-complete and non-κ^+-complete ultrafilters.

In the first result below, we restrict the field k so that we can make use of a theorem of G. Brown [11] or its variant in Bourbaki [10, Ch.VIII, §11, Exercise 13(b)], from either of which it follows that in a finite-dimensional simple Lie algebra over an *algebraically closed field of characteristic* 0, every element is a bracket (not merely a sum of brackets, as it must be in any simple Lie algebra). In §9 we will say more precisely what Brown and Bourbaki prove, and obtain a more general version of this result.

Theorem 10. *Suppose that k is an algebraically closed field of characteristic* 0*, that* $(A_i)_{i\in I}$ *is a family of finite-dimensional simple Lie algebras over* k*, that the index set* I *has cardinality less than any measurable cardinal* $> \mathrm{card}(k)$*, and that* $f: A=\prod_{i\in I} A_i \to B$ *is a surjective homomorphism to a finite-dimensional Lie algebra* B*.*

Then B *is semisimple, and* f *factors as* $\prod_I A_i \to A_{i_1}\times\dots\times A_{i_n}\cong B$*, where the arrow is the projection onto the product of a finite subfamily of the* A_i*. (In particular,* f *splits, i.e., is right-invertible.)*

Proof. By the result of Brown and Bourbaki cited above, in each A_i, every element is equal to a single bracket. Hence the same is true in $A=\prod_I A_i$, and hence in the homomorphic image B of A; so in particular, B is idempotent: $B=[B,B]$.

Now if B is trivial (0-dimensional), the desired result holds vacuously with $n=0$, so assume the contrary. As a nontrivial idempotent Lie algebra, B must have a homomorphism onto a simple Lie algebra C. By Proposition 8, the composite map $A\to B\to C$ must factor through the projection of $A=\prod_I A_i$ onto an ultraproduct $A/\mathcal{U}$, for some ultrafilter $\mathcal{U}$ on I.

By our assumption on the cardinality of I, $\mathcal{U}$ cannot be a nonprincipal $\mathrm{card}(k)^+$-complete ultrafilter. If it were nonprincipal and not $\mathrm{card}(k)^+$-complete, then the field $K=k^I/\mathcal{U}$, over which $A/\mathcal{U}$ is an algebra, would be uncountable-dimensional over k by Theorem 47, so by Proposition 9, the algebra $C/Z(C)=C$ would acquire a structure of algebra over K. Hence C would be uncountable-dimensional, contradicting our hypothesis that B is finite-dimensional.

Hence $\mathcal{U}$ must be a *principal* ultrafilter, determined by some $i_1\in I$, and what Proposition 8 then tells us is that the composite $A\to B\to C$ factors through the projection $A\to A_{i_1}$.

Now if we write $A=A'\times A_{i_1}$ where $A'=\prod_{i\in I-\{i_1\}} A_i$, we see that $B=f(A)$ is the sum of the two mutually annihilating subalgebras $B'=f(A')$ and $f(A_{i_1})$. The latter subalgebra, since it does not go to zero under the map $B\to C$, and since A_{i_1} is simple, must be an isomorphic image of A_{i_1}. In particular, it has trivial center. But the intersection of two mutually annihilating subalgebras of a Lie algebra must lie in their centers; so the subalgebras $B'=f(A')$ and $f(A_{i_1})$ have trivial intersection. Hence $B=B'\times f(A_{i_1})$, and since $B=[B,B]$ we must have $B'=[B',B']$.

We now repeat the argument with the map $A'\to B'$ in place of $A\to B$. By induction on the dimension of B, the "left-over" part (which at this first stage we have called B') must, after finitely many iterations, become zero, and we get a description of f, up to isomorphism, as the projection of $A=\prod_I A_i$ onto a finite subproduct $A_{i_1}\times\dots\times A_{i_n}\cong B$.

Such a projection clearly splits, giving the final assertion. □

Let us next examine homomorphisms on a direct product $\prod_I A_i$ of finite-dimensional *solvable* Lie algebras. We cannot expect that such homomorphisms will in general factor through finite subproducts, since the solvable Lie algebras include the abelian ones, which we noted at the beginning of §2 (under the description "zero-multiplication algebras") can have homomorphisms on direct products showing very unruly behavior. It is nevertheless reasonable to hope that a finite-dimensional homomorphic image of a direct product of solvable Lie algebras will be solvable.

Among finite-dimensional Lie algebras, the solvable ones can be characterized in several ways: When the base field has characteristic 0, they are those admitting no homomorphisms onto simple Lie algebras. In general they are those containing no nonzero idempotent subalgebras, and also those satisfying one of a certain sequence of (successively weaker) identities. Since these conditions do not remain equivalent if one deletes finite-dimensionality, or the assumption that the algebras be Lie, or, in the first case, that k have characteristic 0, the statement we hope to obtain has several possible formulations for general algebras, all of potential interest. Of these, the one in terms of nonexistence of homomorphisms onto simple algebras is ready-made for a proof using Proposition 8. We shall obtain below a result for general algebras in arbitrary characteristic based on that proposition, then as a corollary get the desired statement on finite-dimensional solvable Lie algebras in characteristic 0.

Our method will again require a restriction to avoid complications involving measurable cardinals. In §7, on the other hand, choosing a different condition that translates to "solvable" in the finite-dimensional Lie case, we will get a result which for finite-dimensional Lie algebras yields the same conclusion, without the restrictions on cardinality and characteristic. For general algebras, however, that result does not subsume the result of this section; they are independent.

(The technical reason why the present generalization of solvability will require a cardinality restriction, but the version in §7 will not, is that we have not been able to prove that the property of admitting no homomorphisms onto simple algebras is preserved under countably complete ultraproducts, but we do have the corresponding statement for countable disjunctions of identities, Proposition 49.)

Note that the next result (on not necessarily Lie algebras) is stronger than the above motivation might lead one to expect: the codomain algebra is only required (in the final sentence) to be countable-dimensional, rather than finite-dimensional, and still less is assumed about the dimensionalities of the A_i.

Theorem 11. *Suppose $(A_i)_{i\in I}$ is a family of algebras over an infinite field k, such that no A_i admits a homomorphism onto a simple algebra (or more generally, such that no A_i admits a homomorphism onto a countable-dimensional simple algebra).*

Assume further that, if there are measurable cardinals greater than $\mathrm{card}(k)$, then either $\mathrm{card}(I)$ or the supremum of the dimensions of all the A_i is less than all such cardinals.

Then $\prod_I A_i$ admits no homomorphism onto a countable-dimensional simple algebra.

Proof. Suppose, by way of contradiction, that $f : \prod_I A_i \to B$ is a homomorphism onto a countable-dimensional simple algebra. Lemma 5 and Proposition 8 tell us that for some ultrafilter $\mathcal{U}$ on I, this f factors through the natural map $\prod_I A_i \to \prod_I A_i/\mathcal{U}$.

If $\mathcal{U}$ is principal (in which case it is μ-complete for every cardinal μ), or is nonprincipal but $\mathrm{card}(k)^+$-complete (in which case, $\mathrm{card}(I)$ must be greater than or equal to some measurable cardinal $\mu > \mathrm{card}(k)$, so that the bound on the dimensions of the A_i in the second paragraph of the theorem applies), then Theorem 48 shows that $\prod_I A_i/\mathcal{U}$ is isomorphic to one of the A_i; but by hypothesis, no A_i admits a homomorphism onto B, a contradiction.

On the other hand, if $\mathcal{U}$ is not $\mathrm{card}(k)^+$-complete, we can argue exactly as in the proof of Theorem 10, and conclude that B is uncountable-dimensional, though we assumed the contrary.

So there exists no such f, as was to be proved. □

Here is the resulting statement about solvable Lie algebras. The dimensions of the A_i are still almost unrestricted, but we must make B finite-dimensional to turn the "no simple images" condition into solvability.

Corollary 12. *Suppose $(A_i)_{i\in I}$ is a family of solvable Lie algebras over a field k of characteristic 0, and suppose that, if there exists a measurable cardinal greater than $\mathrm{card}(k)$, then either $\mathrm{card}(I)$ or the supremum of the dimensions of the A_i is less than every such cardinal. (E.g., this is automatic if all A_i are of dimension $\leq$ the continuum; in particular, if they are finite-dimensional.)*

Then any finite-dimensional homomorphic image B of $\prod_I A_i$ is solvable.

Proof. This follows from the preceding theorem, since if a Lie algebra is solvable, it admits no homomorphism onto a simple Lie algebra, and the converse holds in the finite-dimensional case in characteristic 0. □

6. Ultraproducts and almost direct factors.

The arguments of the preceding section were based on reducing the results to be proved to the consideration of homomorphisms from our infinite product onto simple algebras, and applying Proposition 8. But there are situations where that method is not enough. If the A_i are simple non-Lie algebras, we do not have Brown's theorem available to tell us that $\prod_I A_i$ is idempotent, from which we deduced that B had to have a simple homomorphic image. And if, in our consideration of solvable Lie algebras, we either replace solvability by nilpotence, or look at characterizations of solvability applicable in arbitrary characteristic, we again can't use that argument. For such purposes, we would like to have some variant of Proposition 8 not burdened with the condition $Z(B) = \{0\}$ of Lemma 5(iii); perhaps something based on the idea of Lemma 6 rather than Lemma 5.

It would also be nice to replace the assumption of finite-dimensionality of B in the conclusions of both Theorem 10 and Corollary 12 by a more general condition.

We develop below a refinement of Proposition 8 in line with these two ideas. We will use the following concept, motivated by condition (ii′) of Lemma 6.

Definition 13. *For any algebra A, an* almost direct decomposition *of A will mean an expression $A = B + B'$, where B, B' are ideals of A, and each is the two-sided annihilator of the other. In this situation, B and B' will be called* almost direct factors *of A; each will be called the* complementary *factor to the other.*

Remarks: Any algebra A has a smallest almost direct factor, $Z(A)$; its complement, the largest almost direct factor, is A.

If $A = B_1 + B_2$ is an almost direct decomposition, then $Z(B_1) = Z(B_2) = Z(A)$. Indeed, because B_1 is the annihilator of B_2 we have $Z(A) \subseteq B_1$, and hence $Z(A) \subseteq Z(B_1)$; conversely, any $a \in Z(B_1)$ annihilates both B_1 and B_2, hence annihilates A, i.e., lies in $Z(A)$.

If $Z(A) = \{0\}$, the almost direct decompositions of A are its (internal) pairwise direct product decompositions as an algebra. If $Z(A) \neq \{0\}$, an almost direct decomposition is never a direct product decomposition, since the two factors in the decomposition intersect in $Z(A)$. However, an almost direct decomposition of A does induce a direct product decomposition of $A/Z(A)$, as shown in the proof of Lemma 6(iii$'$) $\Longrightarrow$ (ii$'$). On the other hand, not every direct product decomposition of $A/Z(A)$ need arise in that way, as shown by the 3×3 matrix example following that lemma.

An almost direct factor of an almost direct factor of an algebra A is easily seen to be, itself, an almost direct factor of A. If we perform finitely many such successive almost direct decompositions, we get a decomposition $A = B_1 + \dots + B_n$ as a sum of ideals each of which is the annihilator of the sum of the rest. We may call such an expression an almost direct decomposition into several almost direct factors.

The importance for us of almost direct decompositions lies in

Lemma 14. *If $f : A_1 \times A_2 \to B$ is a surjective homomorphism of algebras, or more generally, a homomorphism satisfying $f(A_1 \times A_2) + Z(B) = B$, then $f(A_1) + Z(B)$ and $f(A_2) + Z(B)$ are complementary almost direct factors of B.*

Proof. By assumption, $f(A_1) + Z(B)$ and $f(A_2) + Z(B)$ sum to B, so it remains to prove that they are ideals of B, and are mutual two-sided annihilators. By symmetry, it suffices to show that $f(A_1) + Z(B)$ is an ideal, and is the two-sided annihilator of $f(A_2) + Z(B)$.

Since A_1 is an ideal of $A_1 \times A_2$, its image $f(A_1)$ is an ideal of $f(A_1 \times A_2) + Z(B) = B$, hence so is $f(A_1) + Z(B)$. Since A_1 and A_2 annihilate one another in $A_1 \times A_2$, $f(A_1) + Z(B)$ and $f(A_2) + Z(B)$ annihilate one another in B. So it remains to show that any $b \in B$ which annihilates $f(A_2) + Z(B)$ lies in $f(A_1) + Z(B)$.

Let us write such an element b as $f(a_1) + f(a_2) + z$, with $a_i \in A_i$, $z \in Z(B)$. Since $f(a_1)$ and z automatically annihilate $f(A_2) + Z(B)$, and by assumption $b = f(a_1) + f(a_2) + z$ does, it follows that $f(a_2)$ does. But as a member of $f(A_2)$, it also annihilates $f(A_1)$, hence it annihilates all of $f(A_1) + f(A_2) + Z(B) = B$, i.e., lies in $Z(B)$. Hence the expression $b = f(a_1) + (f(a_2) + z)$ expresses b as a member of $f(A_1) + Z(B)$, as required. ☐

Here is the generalization of finite-dimensionality that we indicated would be helpful in strengthening our results.

Definition 15. *We shall say that an algebra B has* chain condition on almost direct factors *if it has no infinite strictly ascending chain $B_1 \subsetneq B_2 \subsetneq \dots$ of almost direct factors; equivalently, if it has no infinite strictly descending chain $B'_1 \supsetneq B'_2 \supsetneq \dots$ of almost direct factors; equivalently, if it has no infinite chain (totally ordered set) of almost direct factors.*

We do not know much general theory regarding the above chain condition. Clearly, it will hold in a ring with chain condition on two-sided ideals, which is already much weaker than finite-dimensionality. After the main results of this paper, a couple of results on the condition will be proved in §§11.1-11.2. An example in §12.2 will show that not every finitely generated associative algebra over a field satisfies it. Whether every finitely generated Lie algebra over a field does, we do not know.

Here, now, is our modified version of Proposition 8:

Proposition 16. *Suppose $f : \prod_I A_i \to B$ is a surjective homomorphism of algebras, or more generally, a homomorphism such that $f(\prod_I A_i) + Z(B) = B$; and suppose B has chain condition on almost direct factors. Let us abbreviate $\prod_I A_i$ to A, and write $\pi : B \to B/Z(B)$ for the canonical factor map.*

Then there exists a finite family of distinct ultrafilters $\mathcal{U}_1, \dots, \mathcal{U}_n$ on I such that, if we write φ_m for the natural homomorphism $A \to A/\mathcal{U}_m$ $(m = 1, \dots, n)$, then the composite map $\pi f : A \to B/Z(B)$ factors through the map $(\varphi_1, \dots, \varphi_n) : A \to A/\mathcal{U}_1 \times \dots \times A/\mathcal{U}_n$.

Proof. If $B = Z(B)$ this is vacuous, so assume the contrary.

For any partition $I = J \cup (I-J)$, we have the direct product decomposition $A = \prod_J A_i \times \prod_{I-J} A_i$; so by Lemma 14, $f(\prod_J A_i) + Z(B)$ is an almost direct factor of B, with complementary almost direct factor $f(\prod_{I-J} A_i) + Z(B)$. Inclusions of subsets J give inclusions of almost direct factors $f(\prod_J A_i) + Z(B)$, so by our assumption of chain condition on such factors, there must exist some $J_1 \subseteq I$ which yields an almost direct factor B_1 strictly larger than $Z(B)$, but such that every subset $J \subseteq J_1$ yields either the same almost direct factor, B_1, or the trivial almost direct factor, $Z(B)$.

Now by Lemma 14, for every $J \subseteq J_1$ the ideals $f(\prod_J A_i) + Z(B)$ and $f(\prod_{J_1-J} A_i) + Z(B)$ are complementary almost direct factors of B_1; but by our choice of J_1, these can only be B_1 and $Z(B)$ in one or the other order. We claim that the set

(3) $$\mathcal{U}_1^{(0)} \;=\; \{J \subseteq J_1 \mid f(\textstyle\prod_J A_i) + Z(B) = B_1\}$$

is an ultrafilter on J_1. Indeed, the class of subsets of J_1 with the reverse property, $f(\prod_J A_i)+Z(B) = Z(B)$, is closed under pairwise unions and passage to subsets, so $\mathcal{U}_1^{(0)}$ is closed under pairwise intersections and enlargements, i.e., is a filter. Clearly $\emptyset \notin \mathcal{U}_1^{(0)}$, so this filter is proper. And we have noted that for every $J \subseteq J_1$, one of J or $J_1 - J$ is in $\mathcal{U}_1^{(0)}$, so it is an ultrafilter.

Having found an ultrafilter that roughly describes the behavior of f as a map from $\prod_{J_1} A_i$ into the almost direct factor $B_1 = f(\prod_{J_1} A_i) + Z(B)$ of B, we now look at the complementary factor $\prod_{I-J_1} A_i$ of A, which f maps into the complementary almost direct factor $f(\prod_{I-J_1} A_i) + Z(B)$ of B. If the latter is not $Z(B)$, we can repeat the above process with this map. (It was to make this work that we put the "*or more generally*" clause into the first sentence of this proposition and the preceding lemma. If f was assumed surjective to B, this would not guarantee that its restriction to $\prod_{I-J_1} A_i$ would be surjective to $f(\prod_{I-J_1} A_i)+Z(B)$.) Thus we get a subset $J_2 \subseteq I-J_1$ such that $f(\prod_{J_2} A_i)+Z(B)$ is a minimal nontrivial almost direct factor of $f(\prod_{I-J_1} A_i) + Z(B)$, and an ultrafilter $\mathcal{U}_2^{(0)}$ on that subset such that every member of $\mathcal{U}_2^{(0)}$ induces that same almost direct factor.

Iterating this process, we get a strictly decreasing sequence of almost direct factors of B associated with the sets $I,\ I-J_1,\ I-J_1-J_2,\ \dots$; so our chain condition insures that this iteration cannot continue indefinitely. Thus, at some stage, say the n-th, our complementary almost direct factor must be $Z(B)$, so the factor whose complement it is must be the whole algebra we are considering at that stage. Thus, without loss of generality we may, at that stage, take $J_n = I - J_1 - \dots - J_{n-1}$ (rather than some proper subset thereof), giving us a partition $I = \bigcup_{m=1,\dots,n} J_m$; and we see that the ideals $f(\prod_{J_m} A_i) + Z(B)$ $(m = 1,\dots,n)$ constitute an almost direct decomposition of B.

Now for $m = 1,\dots,n$, let $\mathcal{U}_m$ be the ultrafilter on I induced by the ultrafilter $\mathcal{U}_m^{(0)}$ on J_m, i.e., $\mathcal{U}_m = \{J \subseteq I \mid J \cap J_m \in \mathcal{U}_m^{(0)}\}$. For each such m we define a homomorphism

(4) $$g_m :\ A/\mathcal{U}_m \ \to\ B/Z(B)$$

as follows. Any element of $A/\mathcal{U}_m$ is the image of some $a = (a_i)_{i\in I} \in A = \prod_I A_i$. Let us map this by restriction to $\prod_{J_m} A_i$, and then by inclusion into A; this means replacing the components a_i at indices $i \notin J_m$ by zero, while keeping the components at indices $i \in J_m$ unchanged. Map the resulting element by f into B, and then by π into $B/Z(B)$.

If we had chosen a different representative $a' = (a'_i)_{i\in I} \in A$ of our element of $A/\mathcal{U}_m$, then after restriction to J_m, this would have differed from $(a_i)_{i\in I}$ only on a subset of J_m that is not in $\mathcal{U}_m^{(0)}$. But by our construction of $\mathcal{U}_m^{(0)}$, elements with support in such a subset of J_m are mapped into $Z(B)$ by f; so the image under πf of the element obtained from a' equals the image under πf of the element obtained from a, showing that we have described a well-defined map (4).

It is now routine to verify that g_m is a homomorphism $A/\mathcal{U}_m \to B/Z(B)$, with image in $(f(\prod_{J_m} A_i) + Z(B))/Z(B)$, so that the images of $g_1\varphi_1,\ \dots,\ g_n\varphi_n$ annihilate one another; and that $\pi f = g_1\varphi_1 + \dots + g_n\varphi_n$, so that this map indeed factors through $(\varphi_1,\dots,\varphi_n)$. □

The above result says that under the indicated hypotheses we can, in a weak sense, approximate $f : A \to B$ "modulo $Z(B)$" by a homomorphism that factors through $A/\mathcal{U}_1 \times \dots \times A/\mathcal{U}_n$. It is natural to ask whether we can do so in a stronger sense, namely whether we can express f as a "perturbation", of the sort described by Lemma 4, of a genuine homomorphism $f_1 : A \to B$ factoring through $A/\mathcal{U}_1 \times \dots \times A/\mathcal{U}_n$.

We can do this easily if the ultrafilters $\mathcal{U}_m$ are principal: if each $\mathcal{U}_m$ is the principal ultrafilter determined by $i_m \in I$, one finds that the desired $f_1 : A \to B$ can be obtained by projecting A to $A_{i_1} \times \dots \times A_{i_n}$ regarded as a subalgebra of A, and then mapping by f into B.

For nonprincipal $\mathcal{U}_m$, we do not know whether such a factorization is always possible; but we shall show that it is whenever k is a field. Note that what we want is to perturb the given homomorphism f to a homomorphism f_1 whose kernel contains the kernel of the natural surjection $(\varphi_1, \dots, \varphi_n) : A \to A\,/\,\mathcal{U}_1 \times \dots \times A\,/\,\mathcal{U}_n$. The following is a general result on when a homomorphism of algebras over a field has a perturbation whose kernel contains a prescribed ideal.

Lemma 17. *Let k be a field, $f : A \to B$ any homomorphism of k-algebras, and C an ideal of A. Then the following conditions are equivalent:*

(i) *There exists a homomorphism $f_1 : A \to B$ having C in its kernel, such that $f - f_1$ is $Z(B)$-valued.*

(ii) *$f(C) \subseteq Z(B)$, and $f(AA \cap C) = \{0\}$.*

Moreover, if $B = f(A) + Z(B)$, then the first condition of (ii) *is implied by the second.*

Proof. To get (i) $\Longrightarrow$ (ii) (which does not require the assumption that k is a field), suppose we have an f_1 as in (i). Then C is carried into $Z(B)$ by $f - f_1$, but annihilated by f_1, hence it must be carried into $Z(B)$ by f, giving the first assertion of (ii). Further, Lemma 4 (with $h = f_1 - f$) tells us that $f_1 - f$ annihilates AA; and by assumption, f_1 annihilates C, so $f = f_1 - (f_1 - f)$ must annihilate $AA \cap C$, giving the second assertion.

Conversely, assuming (ii), the second assertion thereof shows that the zero map and $-f$ agree on $AA \cap C$, whence there exists a unique k-linear map $h : AA + C \to B$ that agrees with the zero map on AA and with $-f$ on C; and by the first condition of (ii), it will be $Z(B)$-valued. If k is a field, we can extend this as a vector space map (in an arbitrary way) to a map $h : A \to Z(B)$. Since h annihilates AA, Lemma 4 tells us that $f_1 = f + h$ is a k-algebra homomorphism; and since h agrees with $-f$ on C, $f + h$ has C in its kernel.

For the final statement (which again does not require that k be a field), note that to show that $f(C) \subseteq Z(B)$ is to show that $f(C)$ is annihilated on each side by B, which, if $B = f(A) + Z(B)$, is equivalent to being annihilated on each side by $f(A)$. But multiplication on either side by $f(A)$ carries $f(C)$ into $f(AC) + f(CA) \subseteq f(AA \cap C)$, which is zero assuming the second condition of (ii). ☐

Corollary 18. *Under the hypotheses of Proposition 16, if k is a field, then f can be written $f_1 + f_0$, where f_1 is a homomorphism $A \to B$ factoring through $(\varphi_1, \dots, \varphi_n) : A \to A\,/\,\mathcal{U}_1 \times \dots \times A\,/\,\mathcal{U}_n$, and f_0 is a homomorphism $A \to Z(B)$ (necessarily factoring through the natural map $A \to A/AA$).*

Proof. It is easy to see that $(\varphi_1, \dots, \varphi_n)$ maps A surjectively to $A\,/\,\mathcal{U}_1 \times \dots \times A\,/\,\mathcal{U}_n$; hence a homomorphism on A can be factored through that map if and only if its kernel contains $\ker(\varphi_1) \cap \ldots \cap \ker(\varphi_n)$. Hence, by Lemma 17 (including the final sentence), with C taken to be that intersection of kernels, it suffices to show that any a belonging both to $\ker(\varphi_1) \cap \dots \cap \ker(\varphi_n)$ and to AA is in $\ker(f)$.

Given such an a, let $J = \{i \in I \mid a_i = 0\}$. Thus, letting $A_1 = \prod_J A_i$ and $A_2 = \prod_{I-J} A_i$, we have $A = A_1 \times A_2$, and a has zero component in the first factor. Hence since $a \in AA = A_1A_1 + A_2A_2$, we have $a \in A_2A_2$.

Also, since a lies in the kernels of all φ_m, its support $I - J$ belongs to none of the $\mathcal{U}_m$. Hence A_2, which is also supported on that set, is likewise contained in the kernels of all the maps φ_m; so by the conclusion of Proposition 16, $A_2 \subseteq \ker(\pi\, f)$. This says that $f(A_2) \subseteq Z(B)$; hence $f(a) \in f(A_2A_2) \subseteq Z(B)Z(B) = \{0\}$, as required.

The final parenthetical statement is a case of the general observation that any homomorphism from an algebra A to a zero-multiplication algebra factors through A/AA. ☐

If we now add to Corollary 18 the assumptions that the field k is infinite and the algebra B countable-dimensional, we can again (as in the proof of Theorem 10) use Proposition 9 and Theorem 47 to exclude the case where any of the ultrafilters $\mathcal{U}_m$ are non-card$(k)^+$-complete, getting

Theorem 19. *Suppose $f : A = \prod_I A_i \to B$ is a surjective homomorphism of algebras over an infinite field k, where B is countable-dimensional over k and has chain condition on almost direct factors.*

Then there exist finitely many distinct $\mathrm{card}(k)^+$-complete *ultrafilters* $\mathcal{U}_1, \dots, \mathcal{U}_n$ *on* I *such that, writing* φ_m *for the natural homomorphism* $A \to A/\mathcal{U}_m$ $(m = 1, \dots, n)$, f *can be written* $f_1 + f_0$, *where* f_1 *factors through the map* $(\varphi_1, \dots, \varphi_n) : A \to A/\mathcal{U}_1 \times \dots \times A/\mathcal{U}_n$, *and* f_0 *is a homomorphism* $A \to Z(B)$.

In particular, if $\mathrm{card}(I)$ *is not* $\geq$ *any measurable cardinal* $> \mathrm{card}(k)$, *then each of the* $\mathcal{U}_m$ *is a principal ultrafilter, so* f_1 *factors through the projection of* A *onto the product of finitely many of the* A_i.

If it is merely assumed that none of the dimensions $\dim_k(A_i)$ *is* $\geq$ *a measurable cardinal* $> \mathrm{card}(k)$, *then the algebras* $A/\mathcal{U}_m$ *are, at least, each isomorphic to one of the* A_i. □

(Remark: if k is uncountable, the proof of Theorem 47 shows that for a non-$\mathrm{card}(k)^+$-complete ultrafilter $\mathcal{U}$, the field $k^I/\mathcal{U}$ will have dimension at least $\mathrm{card}(k)$ over k. So in that case, one can get the above result not only for B countable-dimensional, but for B of any dimensionality $< \mathrm{card}(k)$.)

7. Solvable algebras (version 2), and nilpotent algebras.

We can now, as promised, prove a result which, for the finite-dimensional Lie case, gives essentially the same conclusion about homomorphic images of direct products of solvable Lie algebras as Corollary 12, but without the restrictions on $\mathrm{card}(I)$ and $\mathrm{char}(k)$, while for general algebras, it is independent of that result. We shall similarly prove the analogous result for *nilpotent* algebras.

Our conditions on general algebras will use the following analogs of the derived series and the lower central series of a Lie algebra. (We modify slightly a common notation for the latter, to avoid confusion with the subscripts indexing the factors in our direct products.)

Definition 20. *In any algebra* A, *we define, recursively,* k*-submodules* $A^{(d)}$ $(d = 0, 1, \dots)$ *and* $A_{[d]}$ $(d = 1, 2, \dots)$ *by*

(5) $$A^{(0)} = A, \qquad A^{(d+1)} = A^{(d)}\, A^{(d)},$$

(6) $$A_{[1]} = A, \qquad A_{[d+1]} = A\, A_{[d]} + A_{[d]} A.$$

We will call A solvable *if* $A^{(d)} = \{0\}$ *for some* d, *and* nilpotent *if* $A_{[d]} = \{0\}$ *for some* d.

(The above concept of nilpotence of a general algebra is standard. That of solvability is less so, but it appears in [32, p.17]. It is not hard to show that the submodules $A_{[d]}$ are in fact ideals, and that the $A^{(d)}$ are subalgebras, and in the Lie case are ideals as well; but we shall not need these facts here.)

Theorem 21. *Suppose* k *is an infinite field, and* $(A_i)_{i \in I}$ *is a family of solvable* k*-algebras, in the sense of Definition 20* (*e.g., solvable Lie algebras in the standard sense*). *Then any finite-dimensional homomorphic image of* $\prod_I A_i$ *is solvable.*

Proof. Say $f : A = \prod_I A_i \to B$ is a homomorphism onto a finite-dimensional algebra. Since finite-dimensionality implies chain condition on almost direct factors, Theorem 19 shows that B is a sum of finitely many mutually annihilating homomorphic images of algebras $A/\mathcal{U}_m$, where the $\mathcal{U}_m$ are $\mathrm{card}(k)^+$-complete ultrafilters on I, together with a subspace $f_0(A) \subseteq Z(B)$.

The $\mathcal{U}_m$ are, in particular, countably complete, and solvability is equivalent to the condition that an algebra satisfy one of the countable family of identities,

(7) $$x = 0, \quad x_0\, x_1 = 0, \quad (x_{00}\, x_{01})\,(x_{10}\, x_{11}) = 0, \quad \dots .$$

Hence by Proposition 49, the condition of solvability on the A_i carries over to the algebras $A/\mathcal{U}_m$. The subalgebra $f_0(A) \subseteq Z(B)$ clearly also satisfies the second identity of (7). Hence, as the identities of (7) are successively weaker, at least one of them will be satisfied by all of the finitely many algebras $A/\mathcal{U}_m$ and by $f_0(A)$.

A sum of finitely many mutually annihilating algebras satisfying a common identity also satisfies that identity, yielding the asserted solvability. □

Exactly the same method yields

Theorem 22. *Suppose* $(A_i)_{i \in I}$ *is a family of nilpotent algebras* (*e.g., nilpotent Lie algebras*) *over an infinite field* k. *Then any finite-dimensional homomorphic image of* $\prod_I A_i$ *is nilpotent.* □

Here, however, a stronger result will be proved in [4], by different methods, with "direct product" generalized to "inverse limit", and no requirement that k be infinite.

(From that result of [4], an analog of Theorem 21 for inverse limits is also deduced, but only for finite-dimensional Lie algebras A_i over a field of characteristic 0.)

8. Simple algebras – general results.

We would now like to use Theorem 19 to get a result on homomorphic images of products of finite-dimensional *simple* Lie algebras stronger than our earlier Theorem 10. Simplicity is not, like solvability or nilpotency, equivalent to a disjunction of identities, but that is not a problem: like countable disjunctions of identities, it is preserved by countably complete ultraproducts (Proposition 50). A more serious difficulty is that the preceding proofs used the fact that $f_0(A)$, a zero-multiplication algebra, was automatically nilpotent and solvable; but if $f_0(A) \neq \{0\}$, it will certainly *not* be a product of simple algebras.

By the final observation of Corollary 18, the map f_0 of Theorem 19 can be nontrivial only if $AA \neq A$, i.e., if A is not idempotent. Simple algebras A_i *are* idempotent. Does this property carry over to direct products?

To answer this, let us define, for every idempotent algebra A, its *idempotence rank,* $\text{idp-rk}(A)$, to be the supremum, over all $a \in A$, of the least number m of summands in expressions for a as a sum of products:

$$(8) \qquad \text{idp-rk}(A) \;=\; \sup_{a\in A}(\inf\{m \geq 0 \mid (\exists\, b^{(1)},\dots,b^{(m)},\, c^{(1)},\dots,c^{(m)} \in A)\;\; a = \textstyle\sum_{h=1}^{m} b^{(h)}c^{(h)}\}).$$

This will be a nonnegative integer (positive if $A \neq \{0\}$), or ∞; it measures the difficulty in asserting the idempotence of A in a uniform way. We can now state and prove

Lemma 23. *For a family of algebras* A_i $(i \in I)$, *the following conditions are equivalent.*

(i) $\prod_I A_i$ *is idempotent.*

(ii) *Every* A_i *is idempotent, and there is a natural number* n *such that for all but finitely many* $i \in I$, $\text{idp-rk}(A_i) \leq n$.

When the above equivalent conditions hold, $\text{idp-rk}(\prod_I A_i) = \sup_{i\in I} \text{idp-rk}(A_i)$.

Proof. We shall prove (ii) $\implies$ (i) and $\neg$(ii) $\implies$ $\neg$(i).

Given n as in (ii), consider any $(a_i)_{i\in I} \in A$. For those i such that $\text{idp-rk}(A_i) \leq n$, take a representation of a_i as a sum of n products, while for each of the remaining finitely many indices i, take *some* representation of a_i as a sum of products. (Some of the A_i may have infinite idempotence rank; but they are all assumed idempotent, so each element a_i can be so written.) There will be a common upper bound N for the number of summands in all these representations, yielding a representation for (a_i) as a sum of N products, proving (i).

Assuming $\neg$(ii), note that if not all A_i are idempotent, then A cannot be. If they all are idempotent, but there is no finite n bounding the idempotence ranks of all but finitely many of them, then it is easy to construct an $(a_i) \in A$ such that the number of products required to express the component a_i is unbounded as a function of i, and to deduce that (a_i) cannot be written as a finite sum of products, proving $\neg$(i).

The verification of the final assertion (which we won't use) is straightforward; one breaks it into two cases, the case where the idempotence ranks of all the A_i are finite, so that (ii) implies that they have a common finite bound, and the case where at least one is ∞. □

Applying this to homomorphic images of direct products of simple algebras, we can now prove

Theorem 24. *Suppose* k *is an infinite field, and* $f : A = \prod_I A_i \to B$ *is a surjective homomorphism from a direct product of simple algebras to a countable-dimensional algebra* B *having chain condition on almost direct factors. Then*

(a) *If there is a finite upper bound on all but finitely many of the values* $\text{idp-rk}(A_i)$ $(i \in I)$, *then* B *is isomorphic to a direct product of finitely many of the* A_i.

(b) *Without the assumption of such a bound,* B *will be isomorphic to a direct product of finitely many of the* A_i, *and one k-vector-space with zero multiplication.*

In the situation of (a), *the homomorphism* f *splits* (*has a right inverse*). *In the situation of* (b), *the composite homomorphism* $A \to B \to B/Z(B)$ *splits.*

Proof. Let f be expressed as in Theorem 19. By Proposition 50, all of the $A/\mathcal{U}_m$ in that description are simple. A homomorphic image of a finite direct product of simple algebras is the direct product of some subset of these; so possibly dropping some of the $A/\mathcal{U}_m$, we may assume that the image of f_1 in B is an isomorphic copy of $A/\mathcal{U}_1 \times \cdots \times A/\mathcal{U}_n$.

Let

(9) $\quad J \ = \ \{i \in I \mid \dim_k(A_i) \text{ is uncountable}\} \ \subseteq \ I.$

If for any of $m = 1, \ldots, n$, the above set J were $\mathcal{U}_m$-large, it is easy to see that $A/\mathcal{U}_m$ would also be uncountable-dimensional. (This does not use the countable completeness of $\mathcal{U}_m$; just the observation that if we had an $\aleph_1$-tuple of k-linearly-independent elements in A_i for each $i \in J$, this would give an $\aleph_1$-tuple of elements of A whose images in $A/\mathcal{U}_m$ would also be linearly independent.) Then $A/\mathcal{U}_m$ could not be embedded in B; so this does not happen. Hence each $A/\mathcal{U}_m$ can be identified with a countably complete ultraproduct of $(A_i)_{i\in I-J}$, a system of countable-dimensional k-algebras. The second paragraph of Theorem 48, with $\mu = \mathrm{card}(k)^+$, now tells us that each $A/\mathcal{U}_m$ is isomorphic to one of the A_i.

In the situation of statement (a), Lemma 23 tells us that A is idempotent, so by the final parenthetical observation of Corollary 18, the f_0 of Theorem 19 is zero. Thus, $B = f_1(A) \cong A/\mathcal{U}_1 \times \cdots \times A/\mathcal{U}_n$, which we have just seen is isomorphic to the direct product of finitely many of the A_i.

In the situation of (b), B will be the sum of $f_1(A)$, as above, and $f_0(A) \subseteq Z(B)$. It is not hard to see that if an algebra B is the sum of a subalgebra $B_0 \subseteq Z(B)$, and a subalgebra B_1 with $Z(B_1) = \{0\}$, then B is the direct product of those two subalgebras, establishing the "direct product" assertion of (b).

To get the final splitting assertion in the situation of (a), note that since $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are finitely many distinct ultrafilters, we can partition I into disjoint sets $J_1, \ldots, J_n$ with $J_m \in \mathcal{U}_m$. Writing $A = \prod_{J_1} A_i \times \cdots \times \prod_{J_n} A_i$, these n factors have pairwise products zero, and the m-th factor maps under f onto the isomorphic image of $A/\mathcal{U}_m$ in B. Thus, f is, up to isomorphism, the direct product of the n canonical maps $\prod_{J_m} A_i \to \prod_{J_m} A_i/\mathcal{U}_m$. As shown in Theorem 48, each of these maps splits; hence so does f.

The situation of (b) is essentially the same, with the composite map $A \to B \to B/Z(B)$ in place of f. $\square$

Note that if we are given a family of idempotent algebras A_i *not* satisfying the condition of (a) above, there always exist homomorphisms f from $A = \prod_I A_i$ onto algebras B for which the zero-multiplication summand of statement (b) is nonzero. For by Lemma 23, A will not be idempotent, hence A/AA will be a nontrivial zero-multiplication homomorphic image of A, and we can take for B any countable-dimensional homomorphic image of A/AA. (A/AA will itself be uncountable-dimensional, for one can partition $(A_i)_{i\in I}$ into infinitely many subfamilies, for each of which the finite-bound condition of (a) fails, and A/AA maps onto the direct product of the zero-multiplication algebras that these yield.)

9. Simple Lie algebras (version 2).

What happens, in particular, if the A_i are finite-dimensional simple Lie algebras? Will we necessarily be in situation (a) of the above theorem?

This comes down to the question of whether, for a fixed base field k, there is a uniform bound on the idempotence ranks of all finite-dimensional simple Lie algebras over k.

At the beginning of §5 we noted that a consequence of a theorem of G. Brown answers that question affirmatively for k algebraically closed of characteristic 0. What Brown in fact proved (in his doctoral thesis, [11]) is that over *any* infinite field k, every classical simple Lie algebra in the sense of Steinberg [35] has (in our language) idempotence rank 1. (The classical simple Lie algebras in that sense comprise both the infinite families $A_n, \ldots, D_n$ and the exceptional algebras, $E_6, \ldots, G_2$.) When k has characteristic 0, these are the *split* simple Lie algebras in the sense of [10], which if k is also algebraically closed are *all* the finite-dimensional simple Lie algebras; so the idempotence ranks of these Lie algebras indeed have a common bound, 1. The Bourbaki reference [10, Ch.VIII, §13, Ex. 13(b)] that we also cited gets the same conclusion, for algebraically closed fields of characteristic 0 only, but with some additional information.

What if k is, instead, the field $\mathbb{R}$ of real numbers? If L is a finite-dimensional simple real Lie algebra, then $L \otimes_{\mathbb{R}} \mathbb{C}$ will be semisimple over $\mathbb{C}$, hence a direct product of one or more simple complex Lie algebras, hence will have idempotence rank 1 by the results quoted. We claim that this implies that L itself has idempotence rank ≤ 2. Indeed, every $a \in L$ can be written within $L \otimes_{\mathbb{R}} \mathbb{C}$ as a bracket $[b+ic,\, d+ie]$ with $b, c, d, e \in L$. Thus, $a = \mathrm{Re}(a) = [b,\, d] - [c,\, e] = [b,\, d] + [-c,\, e]$, a sum of two brackets. (This is noted at [25, Corollary A3.5, p.653], while Theorem A3.2 on the same page shows that every *compact* simple real Lie algebra, i.e., every simple real Lie algebra whose Killing form is negative definite, has idempotence rank 1.) Note that the above argument uses the fact that $\mathbb{C}$ has degree 2 over $\mathbb{R}$. Hence it cannot be extended to give finite bounds on the idempotence ranks of simple Lie algebras over most subfields $k \subseteq \mathbb{C}$, e.g., $\mathbb{Q}$, since $\mathbb{C}$ has infinite degree over these. (The fields for which it works, those over which an algebraically closed field of characteristic 0 has finite degree, which is necessarily 2, are the real-closed fields, the fields that "look essentially like" $\mathbb{R}$.)

However, there is another result in the literature, less obviously related to idempotence rank, that we can use to get what we need in a much wider class of cases. J.-M. Bois [9] proves, using the recently completed classification [31] of finite-dimensional simple Lie algebras L over algebraically closed fields of characteristic not 2 or 3, that every such algebra is generated as a Lie algebra by two elements. We shall show below, first, that such a bound on the number of generators yields something slightly stronger than a bound on the idempotence rank of L, and, then, that for that strengthened version of idempotence rank, change of base field is not a problem; so that from Bois's result on Lie algebras over algebraically closed fields, we can get the result we need for Lie algebras over general infinite fields.

(Notes to the reader of [9]: Though Theorem A thereof does not state the assumption that the base field is algebraically closed, this is clear from the rest of the paper, and Bois (personal communication) confirms this. In [9, §1.2.2], the one part of that paper where non-algebraically-closed base fields k are considered, it is shown that if L is a Lie algebra over an infinite field k such that, on extending scalars to the algebraic closure K of k, the resulting Lie algebra $L \otimes_k K$ can be generated over K by two elements, then L can be so generated over k. But we shall see from examples in §12.6 below that in prime characteristic, a simple L can yield an $L \otimes_k K$ that is not even semisimple, so that [9, §1.2.2] is not applicable to it; and indeed that such an L can fail to be generated by 2 elements. Nevertheless, the ideas of [9, §1.2.2] will be used in proving Theorem 26 below, which shows that any such simple Lie algebra has idempotence rank ≤ 2.)

The fact which turns statements about numbers of generators into statements relevant to idempotence rank, part (c) of the next lemma, would be trivial if we were considering associative algebras. It takes a bit more work in the Lie case, where we must use the Jacobi identity instead of associativity, and is false in general nonassociative algebras (§11.2 below, last sentence). Statements (a) and (b) are steps in the proof that seemed worth recording. These results do not require the base ring to be a field, so we give them for general k.

Lemma 25. *Let L be a Lie algebra over a commutative ring k.*

(a) *If U is a k-submodule of L, then $\{x \in L \mid [x, L] \subseteq U\}$ is a Lie subalgebra of L.*

(b) *If V is a k-submodule of L that generates L as a Lie algebra, then $[V, L] = [L, L]$.*

(c) *If $[L, L] = L$, and L is generated as a Lie algebra by a set X, then $L = \sum_{x \in X} [x, L]$.*

Proof. In (a), the fact that the indicated set is closed under the k-module operations follows from the fact that U is, while closure under Lie brackets comes from the Jacobi identity: if $[x, L]$ and $[y, L]$ are contained in U, then

$$(10) \qquad [\,[x, y], L] \ \subseteq\ [x, [y, L]\,] + [y, [x, L]\,] \ \subseteq\ [x, L] + [y, L] \ \subseteq\ U.$$

To get (b), we apply (a) with $U = [V, L]$. The Lie subalgebra described in (a) then contains V, hence, as V generates L, it equals L. This means that $[L, L] \subseteq [V, L]$; the opposite inclusion is clear.

To get (c), we apply (b) with V the k-submodule spanned by X, and use the assumption $[L, L] = L$ to replace $[L, L]$ in (b) by L. □

(One can prove, more generally, a version of the above lemma for the action of L on an L-module M. E.g., (c) then takes the form, "If $LM = M$ and L is generated as a Lie algebra by a set X, then $M = \sum_{x \in X} x\,M$.")

If we apply (c) to the case where L can be generated by n elements, the resulting conclusion,

$$(11) \qquad (\exists\, x_1, \dots, x_n \in L)\ \ L \ =\ [x_1, L] + \dots + [x_n, L],$$

is formally stronger than the statement that L has idempotence rank n : the idempotence rank statement allows both arguments in the brackets giving an element $a \in L$ to vary as we vary a, while (11) fixes one argument in each bracket. To see that it is strictly stronger, recall that by Brown's result, many finite-dimensional Lie algebras over fields have idempotence rank 1; but no nonzero finite-dimensional Lie algebra over a field can satisfy $(\exists\, x_1)\ L = [x_1, L]$, since any x_1 has nontrivial centralizer, so that $[x_1, L]$ has smaller dimension than L.

Using the above lemma, Bois's Theorem A, and a density argument, we can now prove

Theorem 26. *Let k be an infinite field of characteristic not 2 or 3, and L a finite-dimensional simple Lie algebra over k. Then there exist $x_1,\, x_2 \in L$ such that $L = [x_1, L] + [x_2, L]$.*

Proof. Let K be the algebraic closure of k, and $L_K = L \otimes_k K$. This will be a finite-dimensional Lie algebra over K, and will inherit from L the property of being idempotent; hence as a K-algebra it will have a finite-dimensional simple homomorphic image M. Let $q : L_K \to M$ be the canonical surjection.

Since k is infinite, $L \times L$ is Zariski-dense in $L_K \times L_K$. (I.e., if we represent elements of the finite-dimensional K-vector-space $L_K \times L_K$ in terms of coordinates in some K-basis, then any polynomial function of those coordinates which vanishes on the subset $L \times L$ vanishes everywhere.) It follows that its image $q(L) \times q(L) \subseteq M \times M$ is Zariski-dense in the latter space. In particular, $q(L)$ is nonzero, so, since L is simple, q embeds L in M. In what follows, we shall identify L with $q(L)$.

By Bois [9, Theorem A], M can be generated as a Lie algebra over K by two elements. Moreover, as noted in [9, §1.2.2], the set of generating pairs of elements of M will be a Zariski-open subset of $M \times M$. (I.e., for every generating pair $(x_1,\, x_2) \in M \times M$, there is a finite family of polynomials in the coordinates of x_1 and x_2 which are nonzero at that pair, and such that every pair at which these polynomials are nonzero is again a generating pair. Roughly, this is because any Lie algebra expression $f(x_1, x_2)$ has coordinates given by polynomials in the coordinates of x_1 and x_2, and the property that a given list of $\dim_K(M)$ such expressions spans M over K is equivalent to the nonvanishing of an appropriate determinant in the resulting coordinate polynomials. Cf. [5, §1].) The nonempty Zariski-open set of generating pairs must meet the Zariski-dense set $L \times L$, which means that there exist $x_1,\, x_2 \in L$ which generate M as a Lie algebra over K. Hence by Lemma 25(c),

(12) $\quad M \;=\; [x_1,\, M] \,+\, [x_2,\, M].$

We claim that this implies

(13) $\quad L \;=\; [x_1,\, L] \,+\, [x_2,\, L].$

To show this, let B be a basis of K as a k-vector-space. Then the Lie algebra $L_K = L \otimes_k K$, under the adjoint action of its sub-k-algebra L, is the direct sum of the sub-L-modules $L \otimes b$ $(b \in B)$, each of which is isomorphic to L as an L-module, via the map $x \mapsto x \otimes b$. Since L is simple as a Lie algebra, it is a simple module over itself under the adjoint action, hence since L_K is a direct sum of copies of that simple L-module, so is its homomorphic image M. As M is a direct sum of simple L-modules, L is a direct summand therein. Applying to (12) an L-module projection of M onto L, we get (13), as required. □

For some results on particular elements x_1 and x_2 in split simple Lie algebras L such that $L = [x_1, L] + [x_2, L]$, and related matters, see [29].

Theorem 24(a) and Theorem 26 together give the desired result on infinite products of simple Lie algebras in characteristics $\neq 2,\, 3$. We also record the weaker statement that follows from Theorem 24(b) for characteristics 2 and 3 (where there is as yet insufficient structure theory to say whether a result like that of [9] holds).

Theorem 27. *Suppose k is an infinite field, and $f : \prod_I A_i \to B$ a surjective homomorphism from a direct product of finite-dimensional simple Lie algebras to a finite-dimensional Lie algebra B. Then*

(a) *If $\mathrm{char}(k) \neq 2$ or 3, B is isomorphic to the direct product of finitely many of the A_i, and the map $f : A \to B$ splits.*

(b) *If $\mathrm{char}(k) = 2$ or 3, one can at least say that B is isomorphic to a direct product of finitely many of the A_i and an abelian Lie algebra. In that case, the composite map $A \to B \to B/Z(B)$ splits.* □

A few notes on the general concept of idempotence rank: By Theorem 8.4.5 of [12], every finite-dimensional *simple associative* algebra has a unit, and so has idempotence rank 1. On the other hand, we will give in §12.4 examples of arbitrarily large finite idempotence rank in *non-simple* finite-dimensional Lie and associative algebras, and in simple finite-dimensional *non*-Lie *non*associative algebras; while in §12.5, we will give an example of a finite-dimensional (non-Lie) algebra whose idempotence rank changes under change of base field. (This is the phenomenon, the possibility of which prevented us from using Brown's result to get Theorem 27 over a general field of characteristic 0.)

10. Continuity in the product topology.

Any infinite product $A = \prod_I A_i$ of sets has a natural topology, the product of the discrete topologies on the A_i. If the A_i have group structures, and $f : A \to B$ is a homomorphism into another group B, it is not hard to show that f is continuous in the product topology on A and the discrete topology on B if and only if it factors through the projection of A onto a finite subproduct $A_{i_1} \times \cdots \times A_{i_n}$.

Indeed, "if" is immediate. To see "only if", note that by the discreteness of B, $\ker(f)$ is open. As an open set containing the identity element, it must contain the intersection of the inverse images of neighborhoods of the identity elements e_i of finitely many of the A_i. So a fortiori, it contains the intersection of the inverse images of the trivial subgroups of those A_i; which is the kernel of the projection to their product, so f factors through that projection.

Let us briefly note when this continuity condition holds in the results of the preceding sections.

It is not hard to see that it can never hold if f involves a factorization through a *nonprincipal* ultrafilter.

When f is the sum of a map f_1 that factors through finitely many of the A_i (corresponding to finitely many *principal* ultrafilters), and a possibly nonzero perturbing map f_0 into $Z(B)$, then the continuity of f depends on the continuity of f_0, which in general cannot be expected: as noted in §1, such maps tend to be "unruly". However, the effect of f_0 disappears if we compose f with the factor map $\pi : B \to B/Z(B)$. Summarizing the consequences of these considerations, we have

Proposition 28. *In the preceding results of this note, the map f will be continuous in the product topology on A and the discrete topology on B (equivalently, will factor through the projection to a finite subproduct of the A_i) in the situations of the following results whenever* $\mathrm{card}(I)$ *is less than every measurable cardinal* $> \mathrm{card}(k)$: *Theorem 10 (where that restriction on I is already assumed), Theorem 24*(a)*, and Theorem 27*(a)*.*

Under the same assumption on $\mathrm{card}(I)$*, the composite map $\pi f : A \to B \to B/Z(B)$ will be continuous in the situations of Theorem 19, Theorem 21, Theorem 22, Theorem 24*(b)*, and Theorem 27*(b)*.*

11. Some tangential notes.

We record here further observations on the material introduced in the preceding pages, which were not needed for the results developed there. (The subsections of this section are independent of one another.)

11.1. **Almost direct factors, and Boolean rings.** Recall that for an *associative unital* algebra A, an almost direct decomposition is the same as a direct product decomposition (because $Z(A) = \{0\}$), and that in this situation, such decompositions are in bijective correspondence with the central idempotent elements of A. The set of such central idempotents, and hence the partially ordered set of almost direct factors of A, forms a Boolean ring. Will the same be true of the partially ordered set of almost direct factors in a general algebra A?

Below, we obtain a positive answer when A is idempotent (or satisfies a slight weakening of that condition), then a counterexample in the absence of that assumption.

Proposition 29. *Let A be an idempotent algebra, or more generally, an algebra satisfying*

(14) $\quad A = AA + Z(A).$

Then the almost direct factors of A form a Boolean ring, with zero element $Z(A)$, unit element A, join given by sums of almost direct factors, and meet given by intersections of such factors. Moreover, an intersection $B \cap C$ of almost direct factors of A is also equal to $BC + Z(A)$ and to $CB + Z(A)$.

Proof. Let us write $a \mapsto \overline{a}$ for the quotient map $A \to A/Z(A)$. Any almost direct decomposition $A = B + B'$ is determined by the induced direct product decomposition $\overline{A} = \overline{B} \times \overline{B'}$ (cf. remarks following Definition 13), hence by the projection operator of $\overline{A}$ onto $\overline{B}$. We shall prove below that under the present hypotheses, the projection operators so induced by any two almost direct decompositions $A = B + B'$ and $A = C + C'$ commute, and that the image $\overline{B} \cap \overline{C}$ of their composite corresponds to an almost direct factor $B \cap C = BC + Z(A) = CB + Z(A)$ of A, with complement $B' + C'$. Now a set of pairwise commuting projection operators (i.e., idempotent endomorphisms) on any abelian group generates a Boolean ring of such operators, with the meet and join of operators e and f (given by ef and $e + f - 2ef$ respectively) corresponding to the intersection and the sum of the image subgroups; so these results will prove our claims.

Given almost direct decompositions $A = B + B'$ and $A = C + C'$, let us multiply these two equations together and add $Z(A)$. By (14), this yields $A = BC + BC' + B'C + B'C' + Z(A)$, which we can rewrite

(15) $$A \,=\, (BC + Z(A)) \,+\, (BC' + Z(A)) \,+\, (B'C + Z(A)) \,+\, (B'C' + Z(A)).$$

(Since A is not assumed associative or Lie, we do not yet know that the summands in (15) are ideals of A, only that they are k-submodules.)

Let us verify first that the decomposition $a = a_{BC} + a_{BC'} + a_{B'C} + a_{B'C'}$ of an element $a \in A$ arising from (15) is unique modulo $Z(A)$. For this, it will suffice to show that in any decomposition of 0,

(16) $$0 \,=\, z_{BC} + z_{BC'} + z_{B'C} + z_{B'C'}$$

into summands in the above four k-submodules, all of these summands must lie in $Z(A)$. Now since B and C are ideals, the term BC in the first summand of (15) is contained in both B and C, hence that summand $BC + Z(A)$ annihilates B' and C', hence annihilates all the summands in (15) other than itself; so if the summand z_{BC} of (16) does not lie in $Z(A)$, i.e., does not annihilate all of A, this can only be because it fails to annihilate the first summand of (15). But all the other summands on the right-hand side of (16) do annihilate the first summand of (15), as does the left-hand term, 0; so z_{BC} must also. This completes the verification that it lies in $Z(A)$; and by the same argument, so do all the terms of (16), as claimed. Hence, passing to quotients modulo $Z(A)$, the decomposition

(17) $$\overline{A} \,=\, \overline{BC} + \overline{BC'} + \overline{B'C} + \overline{B'C'}.$$

is a direct product decomposition of k-modules.

Using again the same kind of reasoning, note that when we decompose an element of A by (15), the component in each summand whose expression in (15) involves B annihilates B', and inversely. Hence given such a decomposition $a = a_{BC} + a_{BC'} + a_{B'C} + a_{B'C'}$, the expression $a = (a_{BC} + a_{BC'}) + (a_{B'C} + a_{B'C'})$ decomposes a into an element annihilating B', i.e., a member of B, and an element annihilating B, i.e., a member of B'. But the decomposition of a coming from the relation $A = B + B'$ is unique up to summands in $B \cap B' = Z(A)$; hence the idempotent endomorphism of $\overline{A}$ given by projection on the first summand in $\overline{A} = \overline{B} + \overline{B'}$ must coincide with the projection of (17) onto the sum of its first and second summands. Similarly, the idempotent endomorphism of $\overline{A}$ arising from the decomposition $A = C + C'$ must be the projection of (17) onto the sum of its first and third summands. These two projections commute, since their product in either order is the projection of (17) onto its first summand.

Since the range of the product of two commuting idempotent endomorphisms of an abelian group is the intersection of their ranges, we have $\overline{BC} = \overline{B} \cap \overline{C}$. Taking inverse images in A, this gives

(18) $$BC + Z(A) \,=\, B \cap C,$$

as claimed; and by symmetry we likewise have $CB + Z(A) = B \cap C$.

The equality (18) shows that $BC + Z(A)$ is an ideal; we must still verify that it is an almost direct factor in A. We claim that it and $(BC' + Z(A)) + (B'C + Z(A)) + (B'C' + Z(A))$ are each other's two-sided annihilators. We have seen that they annihilate each other. On the other hand, by the method of reasoning used immediately after (16), if an element a annihilates $BC + Z(A)$, the first component of a decomposition of a as in (15) lies in $Z(A)$; and since $Z(A)$ also lies in the other three summands of (15), a will lie in the sum of those three summands. So the two-sided annihilator of $BC + Z(A)$ is indeed $(BC' + Z(A)) + (B'C + Z(A)) + (B'C' + Z(A))$. That $BC + Z(A)$ is likewise the two-sided annihilator of that sum is shown in the same way. This completes our proof. □

The above result covers not only the case where A is idempotent, but the opposite extreme, where A has zero multiplication, since then $A = Z(A) = AA + Z(A)$. (In that case, our Boolean ring is trivial.) But let us now show that when $A \neq AA + Z(A)$, the corresponding statement need not hold.

Let $A = \mathbb{R}^2 \times \mathbb{R}$, with multiplication $(v, a) * (w, b) = (0, v \cdot w)$, where $v \cdot w$ is the dot product of vectors in $\mathbb{R}^2$. Then $Z(A) = \{0\} \times \mathbb{R}$, and for every one-dimensional subspace $V \subseteq \mathbb{R}^2$, we have the almost direct decomposition $A = (V \times \mathbb{R}) + (V^{\perp} \times \mathbb{R})$, where $(\ )^{\perp}$ denotes orthogonal complement in $\mathbb{R}^2$. Thus, the almost direct factors lying strictly between $Z(A)$ and A form an infinite set of pairwise incomparable elements $V \times \mathbb{R}$ (though for each such element, only one of the others is its "complementary almost direct factor" as we have defined the term). Hence the partially ordered set of almost direct factors of A is not a Boolean ring.

The above algebra A is, incidentally, associative, since all three-fold products are zero.

11.2. **Weakening the definition of an almost direct decomposition.** In our definition of an almost direct decomposition $A = B + B'$ of an algebra A, the condition that B and B' be ideals can be weakened to say that they are subalgebras. For the latter condition on B says that it is closed under left and right multiplication by B, and since it annihilates B', it is trivially closed under left and right multiplication by that subalgebra; hence it closed under left and right multiplication by $B + B' = A$.

If we ask whether it is enough to assume that B and B' are k-submodules summing to A, each of which is the other's two-sided annihilator, the answer is mixed. If A is associative, we can still conclude that they will be almost direct factors. For since B annihilates B' on both sides, associativity implies that BB does the same; hence it is contained in the annihilator of B', namely B, proving that B is a subalgebra. We get the same conclusion if A is a Lie algebra: the Jacobi identity shows that $[B', [B, B]] \subseteq [B, [B', B]] = \{0\}$, hence $[B, B]$ is contained in the annihilator B of B'.

But for general A, the corresponding statement is false. Indeed, for any k, let A be the k-algebra which is free as a k-module on two elements x and y, with multiplication given by

$$(19) \qquad xy = yx = 0, \qquad xx = y, \qquad yy = x.$$

Then clearly, kx and ky are each other's two-sided annihilators, and sum to A, but are not subalgebras.

Even if one of B, B' is a subalgebra, the other may not be, as we can see by replacing the relation $xx = y$ in (19) by $xx = x$, while leaving the other relations unchanged.

Incidentally, taking $X = \{x\}$ in the algebra defined by (19), we find that the analog of Lemma 25(c) fails (X generates A, but $XA \neq A$), showing that that result does not hold for general nonassociative algebras.

11.3. **"Early" ultrafilters.** Just as many calculus texts come in two versions, "early transcendentals" and "late transcendentals", so the development of §§2-6 has an alternative version, in which we obtain our ultrafilters early, before the "either/or" conditions such as Lemma 5(i) by which we summoned them in our present development.

In such a development, one would associate to any map f from a product of nonempty sets $A = \prod_I A_i$ to a set B the family $\mathcal{F}_f$ of $J \subseteq I$ such that f factors through $\prod_J A_i$. This turns out to be a filter, the largest filter $\mathcal{F}$ such that f factors through $A/\mathcal{F}$. Any filter is an intersection of ultrafilters; let us call the set of ultrafilters containing $\mathcal{F}_f$ the "support" of f. One verifies that f factors through the natural map $A \to \prod_{\mathcal{U} \supseteq \mathcal{F}_f} A/\mathcal{U}$. Finally, bringing in the assumption that the A_i and B are algebras and f a surjective homomorphism, one can use the argument of Proposition 8 to show that if B satisfies the conditions of Lemma 5, then the support of f is a singleton $\{\mathcal{U}\}$, while the argument of Proposition 16 shows that if B satisfies the weaker property of chain condition on almost direct factors, then πf has support in a finite set of ultrafilters.

The proofs of the propositions mentioned used the fact that in a direct product of algebras, elements with disjoint support have trivial product. One might get similar results on direct products of groups (or even monoids) using the fact that in a direct product of these, elements with disjoint supports commute. (This is suggested, of course, by the way the brackets of Lie algebras arise from the noncommutativity of Lie groups.) We leave this for the interested reader to investigate.

A very different "early ultrafilters" approach is taken in [6, §3].

11.4. **On idempotence rank, and related functions.** In examining the properties of the idempotence rank function on idempotent algebras, it is helpful to look at a more general version of that situation. For simplicity, let k be a field. Consider any 4-tuple (A, B, C, m), where A, B and C are k-vector-spaces, and m is a surjective linear map $A \otimes_k B \to C$. (Thus, m gives the same information as a k-bilinear map $A \times B \to C$ whose image spans C.) Let us define

$$(20) \qquad \text{max-rank}(m) \;=\; \sup_{c \in C} \inf_{t \in m^{-1}(c)} \text{rank}(t),$$

where $\text{rank}(t)$ denotes the rank of t as a member of the tensor product $A \otimes_k B$, i.e., the minimum number of decomposable tensors $a \otimes b$ that must be summed to get t. We see that when $A = B = C$ is the underlying vector space of an idempotent k-algebra A, and m the map corresponding to the multiplication of A, then $\text{max-rank}(m)$ is precisely $\text{idp-rk}(A)$.

The function $\text{max-rank}(m)$ has a family resemblance to the function $r(M)$ introduced in [2] for a subspace M of a tensor product $A \otimes_k B$, and defined by

$$(21) \qquad r(M) \;=\; \inf_{t \in M - \{0\}} \text{rank}(t).$$

The contexts of the two definitions are essentially the same: what we are given in each is equivalent to a short exact sequence $0 \to M \to A \otimes_k B \to C \to 0$ of k-vector-spaces, with middle term a tensor product. However, neither of these invariants of that short exact sequence seems to be expressible in terms of the other. (If our vector spaces are finite-dimensional, we can form the dual short exact sequence $0 \to C^* \to A^* \otimes_k B^* \to M^* \to 0$, and look at the same two invariants for it, getting, altogether, four invariants from our original sequence, none of which seems to be expressible in terms of the others.)

As noted in [2], $r(M)$ can decrease, but not increase, under extension of base field; for when we make such an extension, the set of elements over which the infimum of (21) is taken is enlarged, while the rank-function on elements lying in the original tensor product remains unchanged. (If we take for M the kernel of the map $A \otimes_k A \to A$ corresponding to the multiplication operation of an algebra, then a decrease in $r(M)$ from a value > 1 to 1 under base extension from k to K means that from a k-algebra A without zero divisors, we get a K-algebra $A \otimes_k K$ with zero divisors. The reverse cannot happen, of course.)

Since the definition (20) of $\text{max-rank}(m)$ involves both a supremum and an infimum, that function can potentially increase or decrease under base extension. The possibility of its decreasing is what made it impossible for us to go from Brown's result showing that $\text{idp-rk}(L) = 1$ for L a finite-dimensional simple Lie algebra over $\mathbb{C}$ to the corresponding statement for subfields of $\mathbb{C}$. In §12.5 we will see examples of finite-dimensional, idempotent (but non-Lie, nonassociative, non-simple) algebras A whose idempotence ranks do increase and decrease under base extensions. We do not know whether either can happen when A is a simple Lie algebra.

What we used in §8 (in conjunction with the results of [9]), instead of the unsuccessful approach indicated above, was an argument via what might be called the "one-variable-constant idempotence rank function", the least n such that (11) holds. In the case of general nonassociative algebras, where left and right multiplication are not equivalent, we could call the version with the constant factors on, say, the left the "left-constant idempotence rank":

$$(22) \quad l\text{-const-idp-rk}(A) \;=\; \inf\{n \mid (\exists\, x_1, \dots, x_n \in A)\ A = x_1 A + \dots + x_n A\} \;=\; \inf\{\dim_k(V) \mid A = VA\}.$$

This function is examined in [5].

11.5. **Other literature on homomorphisms from infinite products.** Restrictions on homomorphisms from infinite direct products to "small" objects have turned up in other areas of algebra.

In [30, Corollary 9], it is shown that a homomorphic image of a direct power of a finite nonabelian simple group G, if countable, must be finite; general finite groups G with that property are investigated in [7] and [8]. This situation has a similar flavor to that of the present note; e.g., note that simple nonabelian groups satisfy the analogs of the conditions of our Lemma 5. (We remark, however, that the groups characterized in the above papers all have trivial centers. Perhaps if one considers groups with nontrivial centers, analogs of the results of this note showing that homomorphisms $\prod A_i \to B$ acquire stronger properties on composition with the natural map $B \to B/Z(B)$ will turn up.)

An area of investigation with a different flavor begins with the result of [34], that every homomorphism of abelian groups $\mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ factors through the projection onto finitely many coordinates. It can be deduced from this that the same factorization property holds for homomorphisms from any countable product of abelian groups $\prod_{i \in \mathbb{N}} A_i$ to $\mathbb{Z}$; this is expressed by saying that $\mathbb{Z}$ is a *slender* abelian group. More generally, slenderness has been studied in abelian monoids, in modules over general rings, and in objects of general preadditive categories. Note that for these abelian groups, abelian monoids, etc., unlike the algebras of this note and unlike nonabelian groups, any finite family of morphisms can be added; hence in mapping a finite product $A_1 \times \dots \times A_n$ to an object B, one can form sums of homomorphisms $A_i \to B$. Thus, the restrictions that turn out to hold on homomorphisms from infinite products of these objects cannot arise from restrictions on homomorphisms from finite products, like those of our Lemma 5, but must come in in a more mysterious way; roughly, it seems, from completeness-like properties of infinite products, which cannot be duplicated in a "slender" B. Examples of abelian groups that are *not* slender include all abelian groups with torsion, all nonzero injective abelian groups, the additive group of p-adic integers, and, of course, $\mathbb{Z}^{\mathbb{N}}$. For a sampling of work in this area, see, for abelian groups, [21, §94] and [33], for modules, [1] and [20], and for abelian monoids and objects of preadditive categories, [14] and [15].

Related conditions have been considered on nonabelian groups, in some cases again defined in terms of homomorphisms from direct products [22] [23], in others, in terms of homomorphisms from certain completions of free products [24] [17] [18].

We remark that for abelian groups and other structures whose morphisms can be added, the class of slender objects would not change if, in the definition, we restricted attention to *surjective* homomorphisms $\prod A_i \to B$, since if $f: \prod A_i \to B$ is a nonsurjective map witnessing the failure of B to be slender, there is an obvious surjective map $B \times \prod A_i \to B$ which does the same. A similar observation applies to the version of slenderness in nonabelian groups defined using the "complete free products" of [17] – but not to the one defined using direct products [22]. Hence if one defines a condition like slenderness for nonabelian groups, but based on *surjective* maps from direct products, one can expect to find a larger class of examples than the ordinary slender groups, and probably techniques and results close to those of this note; cf. next to last paragraph of §11.3. (It is not clear to us whether the class of groups defined similarly in terms of maps from the "unrestricted free products" of [24], [18] would similarly grow if one imposed this condition only on surjective maps from those groups.)

In [19], some implications among conditions on homomorphisms $A^I \to B$ are studied for algebras A and B in the general sense of universal algebra, the cases of slender abelian groups on the one hand, and of discriminator algebras on the other, being noted.

In most of the works cited above, as in this note, many results depend on whether the cardinality of the index-set I is or is not greater than or equal to an uncountable measurable cardinal. We have, for simplicity, limited the results quoted to in this section to the countable-index-set case, though they are mostly equivalent to the corresponding statements for all index sets smaller than all uncountable measurable cardinals.

12. Examples.

In earlier sections, we noted some examples in passing. Here we give further examples, often lengthier, for which we did not want to interrupt the development of our results.

As in §11, the subsections below are independent of one another. The only dependence on that section is that §12.5 below assumes §11.4 above.

12.1. **Idempotent algebras with $Z(A) \neq \{0\}$.** We noted following Proposition 9 that a *unital* algebra A necessarily satisfies $Z(A) = \{0\}$. Is the same true of *idempotent* algebras – perhaps subject to some additional conditions?

An easy example shows that this need not even hold in finite-dimensional algebras of idempotence rank 1. Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions, let $\mathrm{Im}: \mathbb{H} \to \mathbb{H}$ be the "imaginary part" map, $a + bi + cj + dk \mapsto bi + cj + dk$, and let A be $\mathbb{H}$ under the nonassociative multiplication $x * y = \mathrm{Im}(x)\,\mathrm{Im}(y)$ (where the right-hand side is evaluated using the ordinary multiplication of $\mathbb{H}$). Note that if we call the real and imaginary parts of an element of $\mathbb{H}$ its "scalar" and "vector" components, then $x * y$ has for scalar component the negative of the dot product of the vector components of x and y, and for vector component the cross product of those same vectors. Now it is not hard to see geometrically that for any scalar a and vector $bi + cj + dk$, one can find two vectors with dot product $-a$ and cross product $bi + cj + dk$. This gives the asserted idempotence of our algebra. On the other hand, clearly, $Z(A) = \mathbb{R} \neq \{0\}$.

One can get an infinite-dimensional example, again of idempotence rank 1, that is associative and commutative: Let V be any commutative valuation ring with nondiscrete valuation; thus, its maximal ideal $\mathbf{m}$ is idempotent of idempotence rank 1. Take a nonzero element $x \in \mathbf{m}$, and let $A = \mathbf{m}/x\,\mathbf{m}$. Then A has idempotence rank 1, but the image of x lies in $Z(A)$.

We shall also see in §12.4 below, where we give examples of finite-dimensional idempotent associative algebras A of arbitrarily large finite idempotence rank, that such algebras can have $Z(A) \neq \{0\}$. But, curiously, this cannot happen for finite-dimensional associative algebras of idempotence rank 1, by the following observation.

Lemma 30. *Let A be an associative algebra of idempotence rank 1 which has ascending chain condition on right ideals of the form aA $(a \in A)$ (as will hold, for instance, if k is a field and A is finite-dimensional).*

Then every $x \in A$ satisfies $x \in xA$. Thus, no nonzero element left annihilates all of A, so in particular, $Z(A) = \{0\}$.

Proof. Given $x \in A$, the idempotence-rank-1 hypothesis says that we can write $x = ab$ for $a, b \in A$. Among such expressions, let us choose one which maximizes the right ideal bA. Now b can likewise be factored; say $b = cd$. Thus, the ideal cA contains b, hence it contains bA, hence by our maximality assumption, it equals bA. Thus $b \in cA = bA$; so say $b = be$. Now $x = ab = abe = xe$; so $x \in xA$, as claimed. ☐

12.2. **On the chain condition on almost direct factors.** Our next example will show that a finitely generated (unital or nonunital) associative algebra need not satisfy the chain condition on almost direct factors; and thus that the Boolean ring of central idempotents of a finitely generated unital associative algebra can be infinite.

We begin by constructing a family of unital associative algebras A_i $(i = 0, 1, \dots)$ over any field k. For each i, let A_i be presented by three generators x, y, z, and the infinite family of relations

$$(23) \qquad xy^n z = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{otherwise} \end{cases} \qquad (n = 0, 1, \dots).$$

If we regard (23) as a system of "reduction rules" for expressions in x, y, z, we find that, in the terminology of [3], these have no "ambiguities" (roughly, there is no way to write down a word having subwords $xy^m z$ and $xy^n z$ which "overlap", and so force us to worry whether the two competing reductions may fail to lead ultimately to the same expression). Moreover, application of one of these reduction rules to a word in x, y and z yields at most a shorter word, so the process of recursively applying these rules always terminates. Hence, by [3, Theorem 1.2], each A_i has for k-basis the set of words in x, y, z (including the empty word, 1, since at the moment we are considering unital k-algebras) having no subwords of the form $xy^n z$. In particular, the empty word 1 belongs to this basis, so $1 \neq 0$ in each of these algebras.

Now within the product algebra $\prod_{i=0,1,\dots} A_i$, let x be the element whose coordinate in each A_i is the element $x \in A_i$, define y and z analogously, and let $A \subseteq \prod A_i$ be the nonunital subalgebra generated by these three elements. Then for each n, the element $xy^n z \in A$ will have 1 in the n-th coordinate and 0 in all others, and so be a central idempotent. It follows that the elements xz, $xz + xyz$, $\dots$, $\sum_{m=0}^{n} xy^m z$, $\dots$ constitute an infinite ascending chain of central idempotents, yielding an infinite ascending chain of almost direct factors.

If, instead, we take for A the *unital* algebra generated by these same three elements, we get a finitely generated unital associative algebra whose Boolean ring of central idempotents is infinite.

We remark that in (23), for conceptual simplicity, we used algebras A_i such that the sets $\{n \mid xy^n z = 1\}$ were singletons; but for every subset J of the natural numbers, the same structure result applies to the algebra A_J obtained by setting $xy^n z$ to equal 1 if $n \in J$, and 0 otherwise. Applying the above construction to appropriate families of these algebras, we can get a 3-generator algebra A whose Boolean ring of central idempotents is any countably generated Boolean ring.

12.3. **The need for k to be a field in Lemma 17.** In Lemma 17, assuming k a field, we gave a necessary and sufficient condition for a homomorphism $f : A \to B$ of k-algebras to be approximable modulo $Z(B)$ by a homomorphism $f_1 : A \to B$ annihilating a given ideal $C \subseteq A$. The following example shows that the condition given there fails to be sufficient if, instead, k is any integral domain that is not a field (or more generally, if k is a commutative ring that is not von Neumann regular).

Lemma 31. *Let k be a commutative ring having an element c such that $c \notin c^2 k$.*

Let B be the free k/c^2k-module on one generator x, with the (associative, commutative) multiplication defined by $x^2 = cx$; let A_0 be the free k/c^2k-module on one generator y, with the zero multiplication; let $A = B \times A_0$ as k-algebras, and let $f : A \to B$ be the projection onto the first factor. Then the k-submodule C of A generated by $cx - cy$ is an ideal that satisfies condition (ii) *of Lemma 17, but not condition* (i).

Proof. Observe first that $cx - cy \in Z(A)$, hence the k-submodule C that it generates is indeed an ideal, and its image under f lies in $Z(B)$.

Note also that if an element $d(cx - cy)$ $(d \in k)$ of this ideal lies in AA, then its A_0-component dcy must lie in $A_0 A_0 = \{0\}$, hence since the subrings $A_0 = ky$ and $B = kx$ are isomorphic as k-modules, we also have $dcx = 0$, so the given element $d(cx - cy)$ is zero. Thus $C \cap AA = \{0\}$, so our example satisfies condition (ii) of Lemma 17.

Now suppose there were a homomorphism f_1 as in condition (i) of that lemma. Lemma 4(ii) tells us that $f - f_1$ annihilates AA, which contains $x^2 = cx$, so f_1 agrees with f at cx, i.e., it fixes that element. But by assumption, f_1 annihilates $cx - cy \in C$, so it must take cy to the same value; i.e., $f_1(cy) = cx$.

Now writing $f_1(y) = ax \in B$, the above equation becomes $c(ax) = cx$. Applying this twice, we get $cx = acx = a^2cx = a^2x^2 = (ax)^2 = f_1(y)^2 = f_1(y^2) = f_1(0) = 0$, a contradiction. (Intuitively, the algebra structures on kx and ky are too different for there to be a nice choice of f_1 annihilating $cx - cy$.) ∎

12.4. **Unbounded idempotence rank.** Lemma 23 tells us that a product $A = \prod_I A_i$ of finite-dimensional idempotent algebras will fail to be idempotent if the idempotence ranks of those algebras are unbounded; but we have seen that in most characteristics, finite-dimensional *simple* Lie algebras all have idempotence rank ≤ 2, and even in the remaining two characteristics, one may hope that the the same is true. Can we get any examples of finite-dimensional idempotent algebras with arbitrarily large finite idempotence ranks?

We give below three classes of such examples: for associative algebras, for Lie algebras, and for (nonassociative non-Lie) simple algebras, respectively. (Our descriptions of the first two constructions also record the fact that $Z(A)$ is nontrivial, giving examples mentioned in the paragraph before Lemma 30.)

Lemma 32. *For any field k and positive integer i, let A be the k-algebra with underlying vector space the space $M_i(k)$ of $i \times i$ matrices over k, and multiplication "$*$" expressed, in terms of the ordinary multiplication of these matrices, by $a * b = a\, e_{11}\, b$.*

Then A is an associative idempotent algebra with $\text{idp-rk}(A) = i$.

Here $Z(A)$ is spanned over k by $\{e_{mn} \mid 2 \leq m, n \leq i\}$, hence is nonzero if $i \geq 2$.

Proof. It is easy to verify that for any element e of any associative algebra A_0, the operation $a * b = a\, e\, b$ is again associative. (When e is not a right zero divisor, the resulting algebra A is isomorphic as an algebra to the right ideal $A_0 e \subseteq A_0$, via the map $a \mapsto ae$.) So our A is an associative k-algebra. To verify idempotence, we note that each basis element e_{mn} is a product, $e_{m1} * e_{1n}$.

Now recall that the rank, as a matrix, of any product matrix ST in $M_i(k)$ is less than or equal to each of $\text{rank}(S)$ and $\text{rank}(T)$. Hence a product $a * b = a\, e_{11}\, b$ has rank $\leq \text{rank}(e_{11}) = 1$ as a matrix. Thus, to get a matrix of rank i, such as the identity matrix, we need at least i summands.

To show that i summands always suffice, recall that every matrix of rank 1 can be written uv for some column matrix u and row matrix v. If we embed u as the first column of a matrix $u' \in A$, and v as the first row of some $v' \in A$, we see that $u' * v' = uv$. Since every $i \times i$ matrix w is a sum of at most i rank-1 matrices (e.g., the i matrices that agree in one column with w, and have zeroes everywhere else), every matrix w is the sum of i products in A.

It follows immediately from the definition of our multiplication that the elements e_{mn} with $2 \leq m, n \leq i$ are in $Z(A)$, and it is easy to see that any element not in the span of these elements is sent to a nonzero value either by left or right multiplication in A by e_{11}, giving the asserted description of $Z(A)$. $\square$

Here is the closely related Lie example, though we will not attempt to determine its idempotence rank and total annihilator ideal as precisely as in the above case.

Lemma 33. *For any field k of characteristic not 2, and any integer $i \geq 2$, let A be the k-algebra with underlying k-vector-space the subspace of $M_i(k)$ consisting of all matrices in which the coefficients of e_{11} and of e_{22} sum to zero, and with operation given by*

$$[a, b] \;=\; a\,(e_{11} + e_{22})\, b - b\,(e_{11} + e_{22})\, a. \tag{24}$$

Then A is an idempotent Lie algebra with $i/4 \leq \text{idp-rk}(A) \leq i + 3$. (Thus, taking i sufficiently large, we get arbitrarily large idempotence ranks.)

$Z(A)$ contains all elements $\{e_{mn} \mid 3 \leq m, n \leq i\}$, hence is nonzero if $i \geq 3$.

Proof. By the general observation with which we began the proof of the preceding lemma, $M_i(k)$ is an associative algebra under the multiplication $a * b = a\,(e_{11} + e_{22})\, b$; hence it becomes a Lie algebra under the corresponding commutator bracket operation (24).

It is easy to check that the set of basis elements e_{mn} in which one or both of m, n are ≥ 3 spans a 2-sided ideal in the above associative algebra structure. (The only way a product $a * b$ having such a basis element as a or b can be nonzero is if any index ≥ 3 in that basis element is "facing away from" the factor $e_{11} + e_{22}$ in the definition of our multiplication; hence such an index will survive in every term of the product.) Consequently, in examining the coefficients of e_{11} and e_{22} in a commutator $[a, b] = a * b - b * a$, we can without loss of generality assume that all elements e_{mn} occurring in the expressions for a and b have both subscripts in $\{1, 2\}$. Thus, we are reduced to computing in $M_2(k)$, and there our multiplication is the ordinary multiplication, hence our brackets are ordinary commutator brackets, and we know that the value of any such bracket has trace zero. So the range of our bracket operation on $M_i(k)$ contains only matrices in which the coefficients of e_{11} and e_{22} sum to zero, hence the set of matrices with that property indeed forms a Lie algebra A.

This Lie algebra contains the simple, hence idempotent, Lie subalgebra $\mathrm{sl}_2(k)$, so to show A is idempotent, it will suffice to show that the range of the bracket also contains all matrix units e_{mn} with at least one of $m,\ n \geq 3$. For $n \geq 3$, we have

(25) $$e_{1n} = [e_{11} - e_{22},\ e_{1n}].$$

Elements e_{2n}, e_{m1} and e_{m2} are obtained by obvious variants of this calculation. Finally, if both m and n are ≥ 3, we have

(26) $$e_{mn} = [e_{m1},\, e_{1n}],$$

completing the proof of idempotence.

Every bracket $[a, b]$ is by definition a difference of two matrices each of which, under the ordinary matrix multiplication of $M_i(i)$, has an internal factor $e_{11} + e_{22}$, hence both of which have rank ≤ 2. Thus, $[a, b]$ has rank ≤ 4, so at least $i/4$ summands are needed to get an element of rank i. This gives our lower bound on idp-rk(A).

To get the upper bound, note first that if in (26) we hold n fixed, and taken an arbitrary k-linear combination of the resulting equations for all $m \geq 3$, we get, as a single commutator $[x,\, e_{1n}]$, an arbitrary column in position $n \geq 3$ having top two components zero. Thus, summing $i - 2$ such commutators, we can get any matrix living in the lower right-hand $i - 2 \times i - 2$ block of $M_i(k)$. Linear combinations of the equations (25), and of the three variants mentioned following it, show that with four more commutators, we can fill in everything but the upper left 2×2 block of a general member of A. Since every element of $\mathrm{sl}_2(k)$ is a commutator, we can fill in that block in one more step; so every member of A is a sum of $(i - 2) + 4 + 1 = i + 3$ commutators.

The final sentence of the lemma follows from the observation that the elements e_{mn} with $3 \leq m, n \leq i$ lie in the total annihilator ideal of the associative multiplication $a(e_{11} + e_{22})b$, hence a fortiori in the total annihilator ideal of our Lie bracket. $\square$

The final example of this group, giving nonassociative, non-Lie, finite-dimensional *simple* algebras of unbounded idempotence rank, plays further changes on the idea of a multiplication whose outputs are rank-1 matrices.

Lemma 34. *For any field k and positive integer i, let A be the k-algebra with underlying k-vector-space $k^i \times M_i(k)$, and with multiplication defined as follows:*
– For $u, v \in k^i$, written as row vectors, $u * v$ is the matrix $u^{\mathrm{T}} v \in M_i(k)$, where $^{\mathrm{T}}$ denotes transpose.
– For $S, T \in M_i(k)$, $S * T$ is the vector in k^i whose m-th entry is the $(m - 1)$-st main-diagonal entry of the ordinary matrix product ST. Here we treat subscripts cyclically, so that for $m = 1$, the "$(m - 1)$-st" main-diagonal entry means the i-th.
– Products, in either order, of a member of k^i and a member of $M_i(k)$ are zero.
Then idp-rk$(A) = i$, *and A is simple.*

Proof. In the product of any two members of A, the $M_i(k)$ component will be a matrix $u^{\mathrm{T}} v$ $(u, v \in k^i)$, hence will have rank ≤ 1; so at least i such products must be summed to get elements whose $M_i(k)$ components have rank i. To get an arbitrary element (v, S) as a sum $a_1 * b_1 + \dots + a_i * b_i$, one first selects, as the k^i components of $a_1, b_1, \dots, a_i, b_i$, pairs of vectors having products, under our multiplication, summing to S. In all but one of these pairs, one then takes the $M_i(k)$ components zero, and in the remaining pair, one takes for those components matrices S and T such that $S * T$ is the desired first component v. Thus, idp-rk$(A) = i$.

To show that A is simple, let C be a nonzero ideal. Then C either contains an element with nonzero k^i component, or an element with nonzero $M_i(k)$ component.

In the former case, the square of the element in question will have nonzero $M_i(k)$ component, so in either case C contains an element of the latter sort; say (v, S). Suppose the matrix S has nonzero (m, m') entry, which we may assume without loss of generality is 1.

Let us form the product $(v, S) * (0, e_{m'm})$. This will have $M_i(k)$ component zero; to determine its k^i component, note that the only nonzero main-diagonal component of $S\, e_{m'm}$ is a 1 in the m-th position. So by our description of products $S * T$, the element $S * e_{m'm} \in k^i$ will be the vector f_{m+1} with a 1 in the $(m{+}1)$-st position and zeroes elsewhere. So C contains $(f_{m+1}, 0)$. Squaring this, we get $(0, e_{m+1,m+1})$, and squaring that in turn gives $(f_{m+2}, 0)$. Repeating this process i times (and recalling that in this computation, subscripts are treated cyclically), we get all of $(f_1, 0), \dots, (f_i, 0)$.

Taking linear combinations of these gives all elements $(v, 0)$; multiplying pairs of such elements, and adding together families of i such products, gives all elements $(0, S)$; adding these two sorts of elements we get all of A, completing the proof of simplicity. □

12.5. **Idempotence rank and base change.** We will now give an example showing that the idempotence rank of a finite-dimensional algebra can go down (or up) under extension of base field. However, our example will be non-simple and non-Lie.

We arrived at this example by looking, first, for an example of this sort for the invariant $\text{max-rank}(m)$ of a map $m : A \otimes_k B \to C$ of vector spaces over a field k, defined in (20), a generalization of the idempotence rank of an algebra. We wondered whether we could find such a map m for a general field k, which, when restricted to tensors of rank ≤ 1, would be surjective if k was the complex numbers, but such that over the real numbers, the range would be constrained by inequalities in the coordinates; so that on passing from the reals to the complexes, the value of $\text{max-rank}(m)$ would drop from a larger value to 1. A little thought shows that for this to happen, A and B must each be at least 2-dimensional, so we tried $A = B = k^2$. The tensors of rank ≤ 1 within $k^2 \otimes_k k^2$ can be pictured as the matrices of rank ≤ 1 in $M_2(k)$, a set with three degrees of freedom, suggesting that a linear image of this set in k^3 might have the desired properties.

It turns out that if we map a 2×2 matrix $((a_{mn}))$ to the 3-tuple consisting of its upper-right and lower-left entries, and its trace, this has the desired properties. Indeed, for a 3-tuple (a_{12}, a_{21}, t) of elements of k to arise in this way from a matrix $((a_{mn}))$ of rank ≤ 1, the entries a_{11} and a_{22} of the latter matrix must have product $a_{12}a_{21}$ (to make the determinant $a_{11}a_{22} - a_{12}a_{21}$ zero), and must sum to t (by definition of the trace). But two elements of k having sum t and product $a_{12}a_{21}$ must be the roots of the quadratic polynomial $x^2 - tx + a_{12}a_{21}$. Over the complexes, such a polynomial always has roots, but it will have roots over the reals only when the inequality $t^2 - 4\,a_{12}a_{21} \geq 0$ holds.

To embody this idea in an idempotent algebra, let k be any field of characteristic not 2, and A the k-algebra with underlying vector space k^3, and multiplication

$$(27) \qquad (a, b, c) * (a', b', c') \;=\; (ab',\ ba',\ aa' + bb').$$

Note that the components of the product are the $(1, 2)$ entry, the $(2, 1)$ entry, and the trace, of the 2×2 rank-1 matrix $(a, b)^{\mathrm{T}}\,(a', b')$. It thus follows from the preceding observations that an element $(r, s, t) \in A$ is a product in A if and only if $t^2 - 4\,rs$ is a square in k. Hence, if k is algebraically closed (or even if it is quadratically closed, i.e., if every element of k is a square), we get $\text{idp-rk}(A) = 1$.

Conversely, we see that if k is not quadratically closed, $\text{idp-rk}(A)$ will not be 1. Rather, it turns out that it is 2, since any element (r, s, t) can be written $(r, 0, t) + (0, s, 0)$, and each of these summands satisfies our criterion for being a product in A. In particular, if we construct the above algebra over the field of reals, and then extend scalars to the complexes, the idempotence rank drops from 2 to 1.

If, inversely, we start with a quadratically closed field k, and extend scalars to a non-quadratically closed field $K \supseteq k$, then $\text{idp-rk}(A)$ will increase from 1 to 2. (If k is algebraically closed, we must, of course, take K transcendental to get a proper extension. However, a general *quadratically* closed field k can have finite algebraic extensions K which are not quadratically closed [27, Corollary 7.11(1)].)

(Incidentally, our use of the term "quadratically closed", defined above, follows [27], but is distinct from the usage in [28, p.462, Exercises 8-9], where it means that for each $c \in K$, one of c or $-c$ is a square. Evidently, one definition is modeled on the properties of a subfield of $\mathbb{C}$ closed under taking square roots, the other on the properties of a square-root-closed subfield of $\mathbb{R}$.)

12.6. **Lie examples based on inseparability.** To build examples of simple Lie algebras in positive characteristic which misbehave under change of base field, let us start with examples that don't misbehave. Namely,

(28) Let k_0 be a field of characteristic $p > 0$, and L_0 a finite-dimensional simple Lie algebra over k_0, such that for every extension field k of k_0, the Lie algebra $L_0 \otimes_{k_0} k$ is again simple.

For instance, we can take $L_0 = \mathrm{sl}_n(k_0)$ for any $n \geq 2$ relatively prime to p.

Recall now that if K is a finite *inseparable* extension of a field k, then the commutative ring $K \otimes_k K$ has nilpotent elements. (This is easily seen if K is purely inseparable, and so has an element $x \notin k$ with $x^p \in k :$ then $(x \otimes 1 - 1 \otimes x)^p = x^p - x^p = 0$. To see the general case, recall [28, Proposition V.6.6, p.250] that there will exist a separable subextension $F \subseteq K$ over which K is purely inseparable. By the preceding observation, $K \otimes_F K$ has nilpotent elements; but that ring is a homomorphic image of $K \otimes_k K$, and if a homomorphic image of a finite-dimensional algebra over a field has nilpotents, the original algebra must also have them.) From this we get

Lemma 35. *For k_0 and L_0 as in* (28)*, let $k \subseteq K$ be extension fields of k_0, with K a finite inseparable extension of k, and let $L = L_0 \otimes_{k_0} K$, regarded as an algebra over $k \subseteq K$.*

Then L is a finite-dimensional simple Lie algebra over k, but $L \otimes_k K$ is not semisimple (*i.e., it has a nonzero nilpotent ideal*)*.*

Proof. L is clearly a finite-dimensional Lie algebra over k, and is simple by (28).

Note that $L \otimes_k K = (L_0 \otimes_{k_0} K) \otimes_k K \cong L_0 \otimes_{k_0} (K \otimes_k K)$. Since K is inseparable over k, we can find a nonzero nilpotent element $\varepsilon \in K \otimes_k K$. Thus, $L_0 \otimes \varepsilon K$ is a nilpotent ideal of $L_0 \otimes_{k_0} (K \otimes_k K)$. □

Recall next that in the proof of Theorem 26, we were able to pull the property $L = [x_1, L] + [x_2, L]$ down from the case of a field K to that of an infinite subfield k. The next lemma shows that the condition of being generated as an algebra by two elements, from which we proved that property, cannot be pulled down in that way.

Though as is well-known, any finite separable extension field K of a field k can be generated over k by a single element (the Theorem of the Primitive Element [28, Theorem V.6.6, p.243]), we shall use in our construction the fact this fails arbitrarily badly for inseparable extensions. For instance, if we take for k a pure transcendental extension $k_0(t_1, \dots, t_N)$ of k_0, then the degree-p^N extension $K = k(t_1^{1/p}, \dots, t_N^{1/p})$ cannot be generated over k by fewer than N elements [38, Theorem 8.6.4].

Lemma 36. *Given k_0 and L_0 as in* (28)*, let $d = \dim_{k_0}(L_0)$, let n be any positive integer, and let $k \subseteq K$ be extension fields of k_0, such that K is finite over k, but cannot be generated over k by fewer than $nd+1$ elements. Again, let $L = L_0 \otimes_{k_0} K$, regarded as a finite-dimensional simple Lie algebra over $k \subseteq K$.*

Then L cannot be generated as a Lie algebra over k by fewer than $n+1$ elements.

Proof. Let $B = \{b_1, \dots, b_d\}$ be a k_0-basis for L_0. Then B will likewise be a K-basis for L. Given n elements $x_1, \dots, x_n \in L$, their expressions in terms of that basis will involve $d\,n$ coefficients in K. By assumption, $d\,n$ elements cannot generate K over k, so those coefficients lie in a proper subextension $F \subseteq K$. Thus $x_1, \dots, x_n$ lie in $L_0 \otimes_{k_0} F$, a proper k-subalgebra of $L_0 \otimes_{k_0} K = L$. □

13. Some questions, and some directions for further study.

13.1. **Can our cardinality restrictions be weakened?** In the main results of this paper, we have assumed the field k infinite. Some of those results remain formally true – but become trivial – for finite k : the hypothesis that $\mathrm{card}(I)$ be less than any measurable cardinal $> \mathrm{card}(k)$ then says that I is finite. (Recall that under the definition we are following, $\aleph_0$ is a measurable cardinal.)

In Theorem 11 and Corollary 12, we assumed, slightly more generally, that *either* $\mathrm{card}(I)$ *or* the supremum of the dimensions of all the A_i was less than all measurable cardinals greater than $\mathrm{card}(k)$. What this would say for finite k is that either $\mathrm{card}(I)$ is finite, or the dimensions of the A_i have a common finite bound. Under this assumption, the conclusions of those two results follow easily from Proposition 8 and the well-known fact that any ultraproduct of *finite* algebraic structures (with only finitely many operations), of bounded cardinalities, is isomorphic to one of those structures.

What we would like to know, of course, is

Question 37. *Suppose k is a finite field, and $f : \prod_I A_i \to B$ a homomorphism of k-algebras, with I infinite, and no common finite bound assumed on the k-dimensions of the A_i. Do some or all of the main results of this paper* (*other than Theorem 22*) *have versions valid for this case?* (*Or can some other results in the same spirit be established?*)

We have excluded Theorem 22 because, as mentioned, a result on nilpotent algebras with no condition that k be infinite is proved in [4].

Another sort of size restriction in our results on homomorphisms $\prod_I A_i \to B$ concerned the object B. Here *some* restriction is needed, since if we allowed B to be $\prod_I A_i$, the identity map of that algebra would be a counterexample to most of our results. But it is not clear that the conditions need to be as strong as those we have used. For instance

Question 38. *Does Theorem 19 remain true if we delete the hypothesis that B satisfy chain condition on almost direct factors?*

In [6, Theorem 8(i-ii)] we indeed prove a result like Theorem 19 without the chain condition hypothesis – but having, instead, the restriction that $\mathrm{card}(I)$ be countable if k is countable, and be $< \mathrm{card}(k)$ if k is uncountable. So we want to know whether we can do without either sort of condition.

A result in which the condition on the codomain algebra might be weakened in a different way is Theorem 11 above, where the codomain is assumed both simple (much stronger than having chain condition on almost direct factors) and countable-dimensional, and we do not know whether the latter condition can be dropped. We also don't know, in that case, whether we need the restriction on the size of the algebras A_i or the index-set I relative to measurable cardinals. So we ask

Question 39. *If $(A_i)_{i\in I}$ is a family of algebras over an infinite field k, such that no A_i admits a homomorphism onto a simple algebra, can $\prod_I A_i$ admit a homomorphism onto a simple algebra?*

Let us note that in the existing proof of Theorem 11, the dimension-restriction on the codomain *can* be slightly weakened: Using the full strength of Theorem 47, we see that if k has cardinality $> \aleph_1$, we get the indicated nonexistence result (with the parenthetical generalization in the first sentence appropriately adjusted) not just for homomorphisms onto simple algebras that are countable-dimensional, but onto the larger class of simple algebras of k-dimension $< \mathrm{card}(k)$.

Concerning the hypothesis in that result that the dimensions of the A_i be less than any measurable cardinal $> \mathrm{card}(k)$, we wonder whether one might be able to remove this by showing that simple algebras A_i of such large dimensions can be replaced by simple subalgebras of smaller dimensions, without affecting the desired properties.

In [4, §6], examples are given of homomorphic images B of inverse limits A of nilpotent algebras in which B has various properties that inverse limits of nilpotent algebras cannot themselves have; e.g., an associative example where B contains a nonzero element y such that $y \in ByB$, and a nonassociative example where B contains an element such that $y^2 = y$. However, the analog of Question 39 for inverse limits of nilpotent is open; it is part of [4, Question 22].

13.2. **On idempotence rank.** The next question poses the problem that we skirted by using the result of [9] instead of that of [11].

Question 40 (also asked in [25, pp.652-653] for $k = \mathbb{R}$). *Does every finite-dimensional simple Lie algebra L over an infinite field k have idempotence rank 1 ?*

13.3. **On the chain condition on almost direct factors.** In §12.2 we saw that a finitely generated associative algebra over a field need not have chain condition on almost direct factors. On the other hand, a finitely generated *commutative* associative algebra over a field is Noetherian, and so does have that chain condition.

Question 41. *Does every finitely generated* Lie *algebra over a field have chain condition on almost direct factors?*

13.4. **Idempotence rank, number of generators, and base change.** The algebras of §12.5, whose idempotence ranks could increase and decrease under base change, were neither associative nor Lie. Also, the motivating idea of that example – an image-set which, when the base field is $\mathbb{R}$, is constrained by inequalities, but which is not so constrained when the base field is $\mathbb{C}$ – only seems to lead to examples where the idempotence rank changes by 1. So we ask

Question 42. (a) *Do there exist finite-dimensional* associative or Lie *algebras over an infinite field whose idempotence ranks change under base extension?*

(b) *Do there exist finite-dimensional algebras of any sort over an infinite field whose idempotence ranks change by more than 1 under base extension?*

In [5, §2], examples are given of finite-dimensional (nonassociative, non-Lie) algebras over *finite* fields whose idempotence ranks change by arbitrarily large amounts under base extension, and of an infinite-dimensional *commutative associative* algebra over $\mathbb{R}$ whose idempotence rank goes down (though only by 1) on extension of scalars to $\mathbb{C}$.

In the same vein is

Question 43 (J.-M. Bois, personal communication)**.** *Let k be a finite field, K its algebraic closure, and L a finite-dimensional Lie algebra over k. If $L \otimes_k K$ can be generated over K by two elements, can L be generated over k by two elements?*

For instance, is $\mathrm{sl}_n(k)$ *generated over k by two elements for all $n \geq 2$ relatively prime to* $\mathrm{char}(k)$ *?*

13.5. **Centroids to the rescue for our inseparable-extension examples?** In §12.6, where we constructed simple Lie algebras in positive characteristic that "misbehaved" under base change, the trick was to treat them as having base field k, though they were Lie algebras over a larger field K, which was inseparable over k. As K-algebras, they are well-behaved.

If L is a finite-dimensional simple Lie algebra over a field k, the largest field K to which the Lie structure extends, called the *centroid* of L, consists of the k-linear endomorphisms φ of L which respect all the adjoint maps $[x, -\,]$ $(x \in L)$, i.e., which satisfy $\varphi([x, y]) = [x, \varphi(y)]$ $(x, y \in L)$ [26, p.290]. If $K = k$, L is called a *central* simple Lie algebra. Thus, every finite-dimensional simple Lie algebra over a field is a central simple Lie algebra over its centroid. It is plausible that if we make a point of looking at our Lie algebras as algebras over their centroids, the kind of misbehavior obtained in §12.6 will not occur:

Question 44. *Let L be a finite-dimensional* central *simple Lie algebra over a field k of characteristic not 2 or 3.*

(a) *Can L be generated over k by two elements?*

(b) *Will $L \otimes_k K$ be a direct product of simple Lie algebras over K for all extension fields K of k ?*

(c) *If either of the above questions has a positive answer, does it remain so if rather than assuming L central, we merely assume the centroid of L to be separable over k ?*

13.6. **Characteristics 2 and 3.** Of course, we would like to know

Question 45. *Does the conclusion of Theorem 27*(i) *hold in the excluded characteristics, 2 and 3 ?*

Perhaps, when the structure theory is extended to those last two characteristics, the result of [9] that we used in the proof will also go over, yielding an affirmative answer. On the other hand, a weaker result than that of [9], perhaps asserting generation by 3 or 4 elements rather than 2, might be easier to prove than the optimal result, and might not need the full structure theory.

We give the remaining points of this section as topics to be investigated, rather than formal questions.

13.7. **Variant formulations of solvability and nilpotence.** Of the two versions of our result on homomorphic images of direct products of solvable Lie algebras, we got the first, Corollary 12, using the criterion that a finite-dimensional Lie algebra over a field of characteristic 0 is solvable if and only if it admits no homomorphism onto a simple Lie algebra, while for the second, Theorem 21, we used the characterization of solvability (in any characteristic) by a disjunction of identities. Thus, our results on Lie algebras were obtained as cases of two different results on general algebras.

There are other elegant characterizations of solvability of a finite-dimensional Lie algebra L : in arbitrary characteristic, the condition that L have no nontrivial idempotent subalgebra; in characteristic 0, either the condition that the ideal $[L, L]$ be nilpotent (which is sufficient but not necessary in general characteristic), or that L contain no simple subalgebra (necessary, but not sufficient in general characteristic).

Likewise, finite-dimensional *nilpotent* Lie algebras L can be characterized among finite-dimensional Lie algebras in other ways than the one used in Theorem 22: as those with no nonzero ideals C such that $[L, C] = C$, as those with no nonzero elements x such that x belongs to the ideal generated by $[L, x]$, and as those whose nonzero homomorphic images M all have $Z(M) \neq \{0\}$.

It might be of interest to examine how some of these conditions on general algebras behave under homomorphic images of direct products.

13.8. **Semisimple Lie algebras in positive characteristic.** A Lie algebra L is called *semisimple* if it has no nonzero abelian ideal. If the base field k has characteristic zero, the finite-dimensional semisimple Lie algebras are just the finite direct products of simple Lie algebras, so our results on homomorphic images of direct products of simple Lie algebras imply the corresponding statements for products of semisimple Lie algebras.

When the base field has positive characteristic, a finite-dimensional semisimple Lie algebra need not be a direct product of simple Lie algebras [36, p.133, top paragraph]. In fact, the present authors know nothing about their structure.

In particular, what Lie algebras are homomorphic images of finite-dimensional semisimple Lie algebras? It is conceivable that all are. (By analogy, every finite group G is indeed a homomorphic image of a finite group having no abelian normal subgroup, namely, a wreath product of G with a finite simple group.) If so, then little can be said about homomorphic images of infinite products of such algebras – though something might be said about homomorphisms from infinite products *onto* semisimple Lie algebras.

These seem to be questions requiring expertise in the theory of Lie algebras over fields of positive characteristic. An introduction to the subject is [37].

13.9. **Restricted Lie algebras, and other algebras with additional structure.** For k a field of positive characteristic p, a *restricted* Lie algebra or *p-Lie algebra* over k is a Lie algebra given with an additional operation, $x \mapsto x^{(p)}$, satisfying certain identities which, in associative k-algebras, relate the p-th power map with the k-module structure and commutator brackets [26, §5.7]. (The concept can be motivated by the observation that in characteristic p, the set of derivations of an algebra A is closed, in the associative algebra of k-vector-space endomorphisms of A, not only under the vector space operations and commutator brackets, but also under taking p-th powers.)

Thus, p-Lie algebras are not algebras as we define the term in §1. Of course, they have Lie algebra structures, and one can apply our results to that structure; but a p-Lie algebra may, for instance, be simple under its p-Lie algebra structure without being simple under its ordinary Lie algebra structure. We leave to others the investigation of homomorphic images of infinite products of p-Lie algebras, and, generally, of algebras with additional operations.

14. Appendix: Review of ultrafilters and ultraproducts.

We recall here some standard definitions and notation (e.g., cf. [13, p.211*ff*]).

A *filter* on a nonempty set I means a family $\mathcal{F}$ of subsets of I such that

$$(29)\qquad \begin{array}{l} J_1 \supseteq J_2 \in \mathcal{F} \implies J_1 \in \mathcal{F}, \\ J_1,\ J_2 \in \mathcal{F} \implies J_1 \cap J_2 \in \mathcal{F}. \end{array}$$

In view of the first condition, a filter $\mathcal{F}$ is *proper* (not the set of all subsets of I) if and only if $\emptyset \notin \mathcal{F}$.

A maximal proper filter is called an *ultrafilter*; by Zorn's Lemma, every proper filter is contained in an ultrafilter. It is easy to verify that a proper filter $\mathcal{U}$ is an ultrafilter if and only if for every $J \subseteq I$, either $J \in \mathcal{U}$ or $I - J \in \mathcal{U}$. If $\mathcal{F}$ is a filter (in particular, if it is an ultrafilter) on I, one says that a subset $J \subseteq I$ is *$\mathcal{F}$-large* if $J \in \mathcal{F}$. This does not save much ink, but does help with the intuition of the subject.

If $(A_i)_{i \in I}$ is a family of nonempty sets, and $\mathcal{F}$ is a filter on the index set I, then the *reduced product* $\prod_I A_i \,/\, \mathcal{F}$ is the factor-set of $\prod_I A_i$ by the equivalence relation that identifies elements (a_i) and (a'_i) if $\{i \mid a_i = a'_i\}$ is $\mathcal{F}$-large. If all the A_i are furnished with operations making them groups, k-algebras, etc., then this structure can be seen to carry over to their reduced product, making the natural map $\prod_I A_i \to \prod_I A_i \,/\, \mathcal{F}$ a homomorphism. For any $J \in \mathcal{F}$, it is not hard to see that $\prod_I A_i \,/\mathcal{F} \cong \prod_J A_i \,/\mathcal{F}_J$, where $\mathcal{F}_J = \{J' \in \mathcal{F} \mid J' \subseteq J\}$, a filter on J. (This observation depends on our assumption that all A_i are nonempty.)

A reduced product of objects A_i with respect to an *ultrafilter* is called an *ultraproduct* of the A_i. An ultraproduct $A^I \,/\, \mathcal{U}$ of copies of a single object A is called an *ultrapower* of A. Note that in this situation, the diagonal image of A in A^I maps to an isomorphic copy of A within $A^I/\mathcal{U}$.

It is known that an ultraproduct $\prod_I A_i \,/\, \mathcal{U}$ satisfies every first order sentence s which holds on a $\mathcal{U}$-large subfamily of the A_i; i.e., for which $\{i \in I \mid A_i \text{ satisfies } s\} \in \mathcal{U}$ [13, Theorem 4.1.9(iii)].

For every $i_0 \in I$, the filter of all subsets of I containing i_0 is called the *principal* ultrafilter determined by i_0. The ultraproduct of the A_i with respect to that ultrafilter is, up to isomorphism, just A_{i_0}. If I is finite, these are the only ultrafilters on I; if it is infinite, on the other hand, then its cofinite subsets form a proper filter, so there are ultrafilters containing this filter, the *nonprincipal* ultrafilters.

We note, for perspective, that filters on I correspond to ideals in the Boolean ring of subsets of I, by mapping each filter to the ideal of complements of its members. The ultrafilters correspond to the maximal ideals, which in this case are the same as the prime ideals. More generally, for any family of fields $(k_i)_{i \in I}$, the ideals of $\prod_I k_i$ correspond to the filters on I, each filter $\mathcal{F}$ yielding the ideal of all elements having $\mathcal{F}$-large zero-set; in other words, the kernel of the map from $\prod_I k_i$ to the reduced product $\prod_I k_i \,/\, \mathcal{F}$. Again the ultrafilters correspond to the maximal ideals, and these coincide with the prime ideals. (The statements about Boolean rings are essentially the cases of the statements about products of fields in which all k_i are the two-element field.)

15. Appendix: κ-complete and non-κ-complete ultrafilters.

Definition 46 ([13, p.227]). *Let κ be an infinite cardinal. Then an ultrafilter $\mathcal{U}$ on a set I is said to be* κ-complete *if it is closed under intersections of families of* fewer *than κ members. An $\aleph_1$-complete ultrafilter (i.e., one closed under countable intersections) is called* countably complete.

An infinite cardinal κ is called measurable *if there exists a nonprincipal κ-complete ultrafilter on κ.*

(We follow [13] in this definition. Many authors, restrict the term "measurable cardinal" to the case where κ is uncountable, e.g., [16, p.177]. We shall, rather, explicitly write "uncountable measurable cardinal" when that is intended.)

Note that the definition of an ultrafilter makes it $\aleph_0$-complete (closed under finite intersections), so the weakest completeness condition not automatically satisfied is countable completeness. Note also that $\aleph_0$ is, under the above definition, a measurable cardinal, since there exist nonprincipal ultrafilters on it.

It is known [13, Proposition 4.2.7] that for any nonprincipal ultrafilter $\mathcal{U}$ on any index set I, there is a largest cardinal κ such that $\mathcal{U}$ is κ-complete, and that this will be a measurable cardinal. Moreover, if an uncountable measurable cardinal exists, it must be "enormous" in many respects. In particular, truncating set theory to exclude it and all larger cardinals will leave a smaller set theory that still satisfies the standard axiom system ZFC; hence, if ZFC is consistent, so is ZFC together with the statement that there are no uncountable measurable cardinals, and therefore no nonprincipal countably complete ultrafilters [16, Chapter 6, Corollary 1.8].

If uncountable measurable cardinals μ do exist, then any set I admitting a nonprincipal ultrafilter $\mathcal{U}$ that is μ-complete for such a μ must itself have cardinality at least μ [13, Proposition 4.2.2]. It follows that every element of $\mathcal{U}$ must likewise have cardinality at least μ.

Thus, the reader may prefer to assume that there are no uncountable measurable cardinals, or at least that the products of algebras he or she is interested in will always be indexed by sets I of less than any uncountable measurable cardinality, and so read only the first result below, which concerns *non-κ*-complete ultrafilters. But the subsequent results, about κ-complete ultrafilters, show that even if these occur, things still work out fairly nicely for our purposes!

Note that since the sets not in an ultrafilter $\mathcal{U}$ are the complements of the sets in $\mathcal{U}$, the condition that $\mathcal{U}$ be κ-complete is equivalent to saying that if a family of $< \kappa$ sets $I_\alpha \subseteq I$ has union in $\mathcal{U}$, then at least one of the I_α lies in $\mathcal{U}$.

For κ a cardinal, κ^+ denotes the successor of κ, so that a κ^+-complete ultrafilter is one closed under κ-fold intersections; equivalently, one which always contains some member of a κ-tuple of sets if it contains their union. We follow the standard convention that every cardinal κ is the set of all ordinals of cardinality $< \kappa$, hence is itself a set of cardinality κ. The least infinite cardinal, $\aleph_0$, looked at as the set of natural numbers, is denoted ω.

As mentioned in the preceding section, ultraproducts preserve first-order sentences; hence an ultraproduct of fields is not merely a ring, but a field. Our first result below says that *except* in the case involving large measurable cardinals, a nonprincipal ultrapower of an infinite field k is significantly larger than that field.

Theorem 47. *Let k be an infinite field, let $\kappa = \mathrm{card}(k)$, let $\mathcal{U}$ be a* non-κ^+-complete *ultrafilter on a set I, and let $K = k^I/\mathcal{U}$. Then the dimension $[K : k]$ is uncountable, and is at least* $\mathrm{card}(k)$.

Proof. Since $\mathcal{U}$ is not κ^+-complete, we can take a family of κ sets I_α $(\alpha \in \kappa)$ which are not in $\mathcal{U}$, but whose union is in $\mathcal{U}$. Deleting from each the union of those that precede it in our indexing, we may assume that they are disjoint; and throwing in, as one more set, the complement of their union (which is not in $\mathcal{U}$ because their union *is* in $\mathcal{U}$), we may assume the I_α have union I. (Some I_α may be empty.)

We shall prove first that $k^I/\mathcal{U}$ is transcendental over k. Thus, letting t be a transcendental element, the elements $(t-c)^{-1}$ $(c \in k)$ will be k-linearly independent, proving $[K : k] \geq \mathrm{card}(k)$. If k is uncountable, this makes $[K : k]$ uncountable. On the other hand, for countable k, we shall show that $k^I/\mathcal{U}$ is uncountable, again implying that $[K : k]$ is uncountable.

To show $k^I/\mathcal{U}$ transcendental over k, write k as $\{c_\alpha \mid \alpha \in \kappa\}$, with the c_α distinct, and let $t \in k^I/\mathcal{U}$ be the image of the element $c \in k^I$ which has value c_α everywhere on I_α for each $\alpha \in \kappa$. Any nonzero polynomial $p(x)$ over k has only finitely many roots in k, hence its value at c is zero only on a finite union of the I_α, hence not on a member of $\mathcal{U}$; so $p(t) \neq 0$ in $k^I/\mathcal{U}$, showing that t is transcendental.

On the other hand, suppose k is countable, so that our hypothesis on $\mathcal{U}$ is that it is not countably complete. As above, take a decomposition of I into disjoint sets $I_n \notin \mathcal{U}$ $(n \in \omega)$. We shall show that given any countable list of elements of $k^I/\mathcal{U}$, we can find an element not in that list, proving $k^I/\mathcal{U}$ uncountable.

Let the members of our list be the images in $k^I/\mathcal{U}$ of elements $a^{(0)},\ a^{(1)},\ \dots \in k^I$. Then we can choose an element $c \in k^I$ which disagrees with $a^{(0)}$ at each point of I_0 (because k has more than one element), with both $a^{(0)}$ and $a^{(1)}$ at each point of I_1 (because k has more than two elements), and so forth. We then see that for each n, the set at which c agrees with $a^{(n)}$ is a subset of $I_0 \cup \dots \cup I_{n-1} \notin \mathcal{U}$; hence the element of $k^I/\mathcal{U}$ that c defines is distinct from each member of the given countable list, as claimed. □

When $\mathcal{U}$ *is* κ^+-complete, we have a result of the opposite sort.

Theorem 48. *Let k be an infinite field, let $\kappa = \mathrm{card}(k)$, and let $\mathcal{U}$ be a κ^+-complete ultrafilter on a set I. Then the ultrapower $k^I/\mathcal{U}$ coincides with the natural isomorphic copy of k therein.*

In this situation, if μ is any cardinal $> \kappa$ such that $\mathcal{U}$ is μ-complete, and $(A_i)_{i\in I}$ is a family of k-algebras whose dimensions have supremum $< \mu$, then the ultraproduct $\prod_I A_i\,/\,\mathcal{U}$ is isomorphic as a k-algebra to A_{i_1} for some $i_1 \in I$, and the canonical map $\prod_I A_i \to \prod_I A_i\,/\,\mathcal{U} \cong A_{i_1}$ splits (is right invertible).

Proof. The first paragraph follows from the case of the second where all the A_i are one-dimensional, so it suffices to prove the assertions of the latter paragraph. We note that these are immediate if $\mathcal{U}$ is principal, so let us assume the contrary. In that case, we may take μ to be the *greatest* cardinal such that $\mathcal{U}$ is μ-complete. Thus, μ is a measurable cardinal.

We note first that for λ any cardinal, a λ-dimensional k-algebra can, up to isomorphism, be taken to have underlying vector space $\bigoplus_\lambda k$, and its algebra structure will then be determined by λ^3 structure constants $c_{\alpha\beta\gamma}$ $(\alpha,\beta,\gamma \in \lambda)$, where $c_{\alpha\beta\gamma} \in k$ is the coefficient, in the product of the α-th and β-th basis elements, of the γ-th basis element. (These are subject to the constraint that for each α and β, there are only finitely many γ with $c_{\alpha\beta\gamma} \neq 0$; but for counting purposes, this will not matter to us.) Thus, the number of isomorphism classes of λ-dimensional algebras is $\leq \kappa^{\lambda^3}$. From the fact that a measurable cardinal μ is inaccessible [13, Theorem 4.2.14(i)], it follows that for any $\lambda < \mu$, the cardinality of the set of isomorphism classes of k-algebras of dimension $\leq \lambda$ is also $< \mu$. Thus, under the hypotheses of the second paragraph of our theorem, the A_i fall into $< \mu$ isomorphism classes.

Hence if we partition I according to the isomorphism class of A_i, the μ-completeness of $\mathcal{U}$ implies that the subset I_0 corresponding to some one of these classes belongs to $\mathcal{U}$. Let us assume for notational convenience that all the A_i with $i \in I_0$ are equal, and call their common value $A_{(0)}$. We now define the homomorphism

(30) $\quad \psi:\ A_{(0)} \ \to\ \prod_I A_i, \quad \text{where } \psi(a)_i = a \text{ if } i \in I_0, \quad \psi(a)_i = 0 \text{ otherwise.}$

Composing this with the canonical homomorphism

(31) $\quad \varphi:\ \prod_I A_i \ \to\ \prod_I A_i\,/\,\mathcal{U},$

we clearly get an embedding $\varphi\psi : A_{(0)} \to \prod_I A_i\,/\,\mathcal{U}$.

We claim that this is surjective, and hence an isomorphism. (This is one direction of [13, Proposition 4.2.4], but quick enough to prove directly.) For $A_{(0)}$, being $< \mu$-dimensional, has cardinality $< \mu$, hence given any $a = (a_i)_{i\in I} \in A_{(0)}^{I_0}$, we may partition I_0 into $< \mu$ subsets $I_{0,b} = \{i \in I_0 \mid a_i = b\}$ $(b \in A_{(0)})$. Again, by μ-completeness one of these lies in $\mathcal{U}$; say I_{0,b_0}. It follows that a falls together with $\psi(b_0)$ under φ, proving surjectivity of $\varphi\psi : A_{(0)} \to \prod_I A_i/\mathcal{U}$. Thus, $\varphi\psi$ is an isomorphism, so in particular, φ splits. □

If one deletes the bound assumed above on the dimensions of the A_i, one can still show that $\prod_I A_i\,/\,\mathcal{U}$ has properties very close to those algebras, as illustrated by the next two results. These are instances of [13, Theorem 4.2.11], which says that the fact quoted earlier, that ultraproducts preserve first-order sentences satisfied on $\mathcal{U}$-large sets, can be strengthened, for κ-complete ultrafilters, to refer to an extended first-order language allowing conjunctions and disjunctions of all families of fewer than κ sentences. (Incidentally, in the first result below, we could replace each identity $W_m = 0$ with an arbitrary set of identities; but for simplicity of statement we refer to single identities, the case needed in in §7.)

In the preceding theorem, the assumption that k be a field was not essential, but a wordier statement and proof would have been needed without it. The next two results are equally easy to state and prove for general k, so we revert to our default assumption that k is any commutative ring.

Proposition 49 (cf. [13, Theorem 4.2.11]). *Let $(A_i)_{i\in I}$ be a family of k-algebras, and $\mathcal{U}$ a countably complete ultrafilter on the index set I.*

Suppose $W_1 = 0$, $W_2 = 0$, $W_3 = 0, \dots$ is a countable list of identities for k-algebras, such that each A_i satisfies at least one of these identities. Then the ultraproduct $\prod_I A_i/\mathcal{U}$ satisfies at least one of those identities.

Proof. For $m = 1, 2, \dots$, let J_m be the set of $i \in I$ such that A_i satisfies the identity $W_m = 0$. By assumption, $I = \bigcup_m J_m$, hence since $\mathcal{U}$ is countably complete, there is an m such that $J_m \in \mathcal{U}$. Hence $\prod_{J_m} A_i$ satisfies $W_m = 0$, hence so does its image, $\prod_I A_i/\mathcal{U}$. □

Proposition 50. *Let $(A_i)_{i\in I}$ be a nonempty family of k-algebras, and $\mathcal{U}$ a countably complete ultrafilter on the index set I. Then if all A_i are simple k-algebras, so is the ultraproduct $\prod_I A_i/\mathcal{U}$.*

Proof. An algebra A is simple if and only if (i) $A \neq \{0\}$, and (ii) for every nonzero $a \in A$, the set $aA + Aa$ generates the improper ideal of A. Clearly, (i) carries over from the A_i to their ultraproduct. Note that (ii) means that for every nonzero $a \in A$, every $b \in A$ can be represented as a sum of products of elements of A, all of length ≥ 2, in which each product includes a factor a. (We have made no reference in this statement to coefficients in k. This was our point in using products of length ≥ 2 : After selecting an instance of a in each product, we can absorb a coefficient from k, if any, into one of the *other* factors.)

Now there are only countably many forms such an expression as a sum of products can take. Indeed, to generate a form for such an expression, one chooses the natural number that is to be the number of summands, for each summand one chooses the natural number ≥ 2 that is to be its length, and given this length, one chooses one of the finitely many positions for the factor a to appear in, and, finally, one of the finitely many ways for the (nonassociative) product to be bracketed.

Now let us be given $(a_i) \in \prod_I A_i$ with nonzero image in $\prod_I A_i/\mathcal{U}$, and any $(b_i) \in \prod_I A_i$. Let $J = \{i \in I \mid a_i \neq 0\}$. Since the image of (a_i) in our ultraproduct is assumed nonzero, we have $J \in \mathcal{U}$. For each $i \in J$, the simplicity of A_i says that we can write b_i as a sum of products of lengths ≥ 2 in elements of A_i, such that each of these products has a factor a_i. Choosing for each i such an expression for b_i, we can now partition J as $\bigcup_{m\in\omega} J_m$ according to the form of this expression, since we have seen that there are only countably many such forms. By countable completeness, for some m we have $J_m \in \mathcal{U}$. Suppose our m-th expression involves n variables other than a. Then we can choose n elements of $\prod_I A_i$ which, at every $i \in J_m$, represent the values used in our expression for b_i. The images of these elements in $\prod_I A_i/\mathcal{U}$ will therefore witness the condition that the image of (b_i) lies in the ideal generated by the image of (a_i). This proves the simplicity of $\prod_I A_i/\mathcal{U}$. □

The above result is not true for non-countably-complete ultrafilters. For instance, if k is any field, then within the product of matrix rings $\prod_{i\in\omega} M_i(k)$, the set of elements $(a_i)_{i\in\omega}$ $(a_i \in M_i(k))$ such that the set of integers $\{\mathrm{rank}(a_i) \mid i \in \omega\}$ is *bounded* forms an ideal. (Closure under multiplication by arbitrary elements of $\prod M_i(k)$ is immediate; closure under addition follows from the observation that if $\{\mathrm{rank}(a_i) \mid i \in \omega\}$ is bounded by m and $\{\mathrm{rank}(b_i) \mid i \in \omega\}$ by n, then $\{\mathrm{rank}(a_i + b_i) \mid i \in \omega\}$ is bounded by $m + n$.) It is easy to verify that for any nonprincipal ultrafilter $\mathcal{U}$ on ω, the image of this ideal in $\prod_i M_i(k)/\mathcal{U}$ is a proper nonzero ideal, so unlike the $M_i(k)$, the ultraproduct ring is not simple. (One can show that the ideals of this ring form an uncountable chain.)

This finishes the material required for the preceding sections of this paper. We end by showing, for completeness's sake, how part of the proof of Theorem 47 above can be strengthened.

In that proof, K was an ultrapower of a field k with respect to a non-card$(k)^+$-complete ultrafilter, and we showed the degree $[K : k]$ to be uncountable by different methods depending on whether k was countable or uncountable: in the former case by showing that K had uncountable cardinality; in the latter, by showing that it was transcendental over k. Note that in the former situation, it follows that K has transcendence degree over k equal to its cardinality; but our argument in the latter case only proved transcendence degree ≥ 1. Can we similarly show in the second case that $k^I/\mathcal{U}$ has transcendence degree over k equal to its cardinality?

This is immediate if $\mathrm{card}(k^I/\mathcal{U}) > \mathrm{card}(k)$. To see this, note that it is easy to verify that whenever F is a transcendental extension of a field E, one has

(32) $\mathrm{card}(F) \ = \ \sup(\aleph_0,\ \mathrm{card}(E),\ \mathrm{tr.deg}_E(F)).$

Hence,

(33) If $\mathrm{card}(F) \ > \ \sup(\aleph_0,\ \mathrm{card}(E))$, then $\mathrm{tr.deg}_E(F) \ = \ \mathrm{card}(F)$.

So if $\mathrm{card}(k^I/\mathcal{U}) > \mathrm{card}(k) \geq \aleph_0$, we indeed get $\mathrm{tr.deg}_k(k^I/\mathcal{U}) \ = \ \mathrm{card}(k^I/\mathcal{U})$.

Can the contrary case,

(34) $\mathrm{card}(k^I/\mathcal{U}) \ = \ \mathrm{card}(k)$

occur for a non-$\mathrm{card}(k)^+$-complete ultrafilter $\mathcal{U}$? Yes. For instance, if we choose k so that $\mathrm{card}(k) = 2^{\mathrm{card}(I)}$, then $\mathrm{card}(k^I) = (2^{\mathrm{card}(I)})^{\mathrm{card}(I)} = 2^{\mathrm{card}(I)} = \mathrm{card}(k)$, so $\mathrm{card}(k^I/\mathcal{U})$ certainly can't be larger.

We sketch below a proof that we nevertheless have $\mathrm{tr.deg}_k(k^I/\mathcal{U}) = \mathrm{card}(k^I/\mathcal{U})$, though under a slightly stronger hypothesis that that of Theorem 47; namely, with the condition of that theorem that $\mathcal{U}$ be non-$\mathrm{card}(k)^+$-complete strengthened to "non-countably-complete". (This is strictly stronger only if $\mathrm{card}(k)$ is $\geq$ some uncountable measurable cardinal.)

Proposition 51. *Let k be an infinite field, and $\mathcal{U}$ a non-countably-complete ultrafilter on a set I. Then* $\mathrm{tr.deg}_k(k^I/\mathcal{U}) = \mathrm{card}(k^I/\mathcal{U})$.

Sketch of proof. As noted above, the result is straightforward unless (34) holds, so assume (34).

Let k_0 be any countable (possibly finite) subfield of k (e.g., its prime subfield), and note that since, by (34) and Theorem 47, k is uncountable, (33) shows that $\mathrm{tr.deg}_{k_0}(k) = \mathrm{card}(k)$. Let $\{s_{(\alpha)} \mid \alpha \in \mathrm{card}(k)\}$ be a transcendence basis for k over k_0.

Since $\mathcal{U}$ is not countably complete, we can decompose I into a countable family of disjoint $\mathcal{U}$-small subsets I_n $(n \in \omega)$. For each $\alpha \in \mathrm{card}(k)$, let $t_{(\alpha)}$ be the image in $k^I/\mathcal{U}$ of the element of k^I which on each I_n has the constant value $s^n_{(\alpha)}$. We claim that these $t_{(\alpha)}$ are algebraically independent over k. Briefly, if they satisfied an algebraic dependence relation over k, they would satisfy such a relation over the pure transcendental subfield $k_0(s_{(\alpha)})_{\alpha\in\mathrm{card}(k)}$. Hence, clearing denominators, we would get a polynomial relation among the $t_{(\alpha)}$ with coefficients in the polynomial ring $k_0[s_{(\alpha)}]_{\alpha\in\mathrm{card}(k)}$. If this holds in $k^I/\mathcal{U}$, then the corresponding equation among elements of k^I must hold at points of infinitely many I_n, and by choosing n larger than the degrees in the $s_{(\alpha)}$ of all the coefficients of the given polynomial, one gets a contradiction.

This shows that the transcendence degree of $k^I/\mathcal{U}$ over k is at least $\mathrm{card}(k)$, which, by (34), equals $\mathrm{card}(k^I/\mathcal{U})$. □

We do not know whether in Proposition 51 the assumption that $\mathcal{U}$ is not countably complete can be weakened to "not $\mathrm{card}(k)^+$-complete".

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