Comments, corrections, and related references welcomed, as always!

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HOMOMORPHIC IMAGES OF PRO-NILPOTENT ALGEBRAS

GEORGE M. BERGMAN

ABSTRACT. It is shown that any finite-dimensional homomorphic image of an inverse limit of nilpotent not-necessarily-associative algebras over a field is nilpotent. More generally, this is true of algebras over a general commutative ring k, with "finite-dimensional" replaced by "of finite length as a k-module".

These results are obtained by considering the multiplication algebra $M(A)$ of an algebra A (the associative algebra of k-linear maps $A \to A$ generated by left and right multiplications by elements of A), and its behavior with respect to nilpotence, inverse limits, and homomorphic images.

As a corollary, it is shown that a finite-dimensional homomorphic image of an inverse limit of finite-dimensional solvable Lie algebras over a field of characteristic 0 is solvable.

Examples are given showing that *infinite*-dimensional homomorphic images of inverse limits of nilpotent algebras can have properties far from those of nilpotent algebras; in particular, properties that imply that they are not residually nilpotent.

Several open questions are noted.

1. General definitions, and a preview of the proof.

Throughout this note, k will be a commutative associative unital ring, and an "algebra" will mean a k-algebra; i.e., a k-module A given with a k-bilinear multiplication $A \times A \to A$, not necessarily associative or unital.

Recall that if A is a nonunital *associative* algebra contained in a *unital* associative algebra A', then the identity

(1) $\quad (1+x)(1+y) \ = \ 1 + (x+y+xy) \quad (x, y \in A)$

which holds in A' motivates one to define, on A, the operation of *quasimultiplication*,

(2) $\quad x * y \ = \ x + y + xy.$

This is again associative, and has 0 as identity element; an element $x \in A$ is called *quasiinvertible* if there exists $y \in A$ such that $x * y = y * x = 0$; equivalently, if $1 + y$ is invertible in A'. Every nilpotent element $x \in A$ is quasiinvertible, with quasiinverse $-x + x^2 - \cdots + (-x)^n + \cdots$. The Jacobson radical of A is the largest ideal consisting of quasiinvertible elements; in particular, an associative algebra is Jacobson radical if and only if every element is quasiinvertible. (We shall write "Jacobson radical" and "radical" interchangeably in this note, using the former mainly in statements of results. We shall only use these terms in the reference to associative algebras.)

If A is a not-necessarily-associative algebra, let us write $\mathrm{Endo}(A)$ for the associative unital k-algebra of all endomorphisms of A as a k-module. (Since $\mathrm{End}(A)$ should denote the set of k-algebra endomorphisms, we use a different symbol for the set of module endomorphisms.) For every $x \in A$, one defines the left and right multiplication maps, $l_x,\ r_x \in \mathrm{Endo}(A)$, by

(3) $\quad l_x(y) \ = \ x\,y, \quad r_x(y) \ = \ y\,x,$

and denotes by $M(A)$ the generally *nonunital* subalgebra of $\mathrm{Endo}(A)$ generated by these maps, as x runs over A, called the *multiplication algebra* of A.

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An algebra A is called *nilpotent* if for some $n > 0$, all length-n products of elements of A, no matter how bracketed, are zero. We shall see that $M(A)$ is nilpotent if and only if A is nilpotent (not hard to prove, but not quite trivial).

Hence, if A is nilpotent, the associative algebra $M(A)$ will in particular be radical. Now the property of being nilpotent is not preserved by inverse limits, but that of being radical is, and is likewise preserved under surjective homomorphisms. To use these facts, we have to know how $M(A)$ behaves with respect to homomorphisms and inverse limits.

A homomorphism $h : A \to B$ need not induce a homomorphism $M(h) : M(A) \to M(B)$; but it will if h is surjective, and $M(h)$ will then also be surjective. The need for h to be surjective will not be a problem for us, because if an algebra A is an inverse limit of nilpotent algebras A_i, then by replacing the A_i by appropriate subalgebras, we can get a new system having the same inverse limit A, and such that the new projection maps $A \to A_i$ and connecting maps $A_i \to A_j$ are surjective. Once these conditions hold, one has

$$(4) \qquad M(\varprojlim A_i) \ \subseteq\ \varprojlim M(A_i) \ \subseteq\ \mathrm{Endo}(\varprojlim A_i).$$

Thus, if the A_i are all nilpotent, the elements of $M(\varprojlim A_i)$ will all have quasiinverses in the radical algebra $\varprojlim M(A_i)$, and hence in $\mathrm{Endo}(\varprojlim A_i)$. From this we shall be able to deduce that if B is a homomorphic image of $A = \varprojlim A_i$, then for all $u \in M(B)$, the linear map $1 + u \in \mathrm{Endo}(B)$ is surjective.

If, moreover, B has finite length, this surjectivity makes the maps $1 + u \ \ (u \in M(B))$ invertible; i.e., it makes the elements u quasiinvertible in $\mathrm{Endo}(B)$. If we could say that they were quasiinvertible in $M(B)$, this would make $M(B)$ radical. We can't initially say that; but we shall find that the quasiinvertibility of the images of quasiinvertible elements allows us to extend our map $M(A) \to \mathrm{Endo}(B)$ to a radical subalgebra of $\mathrm{Endo}(A)$ containing $M(A)$. Since a homomorphic image of a radical algebra is radical, we get a radical subalgebra of $\mathrm{Endo}(B)$ containing $M(B)$. Using once more the finite length assumption on B, we conclude that that subalgebra of $\mathrm{Endo}(B)$ is nilpotent, hence so is $M(B)$; hence so is B, yielding our main result (first paragraph of abstract).

One may ask why it is that in the above development, before assuming that B had finite length, we could conclude that the maps $1 + u$ were surjective, but not that they were injective. I don't have a deep answer, but let me sketch a counterexample to injectivity (to be given in detail in §6), then show why surjectivity still holds in that situation.

Suppose one takes an inverse limit A of nilpotent associative algebras A_i, and divides out by the two-sided ideal (r) generated by an element of the form

$$(5) \qquad r \ =\ y \,-\, x\,y\,z \ =\ (1 - l_x r_z)(y),$$

where $x, y, z \in A$. Now – to digress for a moment – if we were instead dividing out by the ideal generated by $s = y - xy = (1 - l_x)(y)$, we could use the fact that $-x$ is quasiinvertible (with quasiinverse which we can think of as $x + x^2 + \dots$, and which indeed arises from elements of that form in the nilpotent algebras A_i), and, formally left multiplying the equation $s = y - xy$ by $1 +$ the quasiinverse of $-x$, conclude that $s + (x + x^2 + \dots)s = y$. Thus, y would belong to the ideal (s) we had divided out by. Similarly, if we had divided out by $t = y - yz$, we would again find that y lay in the resulting ideal. But when the element we are using is (5), the formula that would be needed to recover y from it is $y = r + xrz + x^2rz^2 + \dots$. We shall show that this cannot be obtained by finitely many algebra operations from r, i.e., y does not lie in (r). Hence, in $B = A/(r)$, the element y annihilated by the operator $1 - l_x\, r_z$ is nonzero, so this operator on B is not injective. (From this it follows that B is not residually nilpotent: any homomorphism to a nilpotent algebra must send y to 0.)

What about surjectivity? Given elements x, y and z of any inverse limit A of nilpotent algebras, let us look for an element of A sent to y by the same map $1 - l_x\, r_z$. Formally, such an element should look like $w = y + xyz + x^2yz^2 + \dots$. Though this can't be obtained from y by the algebra operations within A, the corresponding element can be so obtained in each of the nilpotent algebras from which A was constructed; hence the inverse limit construction, applied to these, gives us an element $w \in A$ which will indeed satisfy $y = w - xwz$. That relation will be preserved in any homomorphic image B of A, so the operation on $M(B)$ given by $1 - l_x\, r_z$ continues to be surjective.

I am indebted to Nazih Nahlus for conjecturing the main result of this note in the case where the A_i are finite-dimensional Lie algebras over an algebraically closed field of characteristic 0, and for pointing out that

that result implies Corollary 19 below. For some parallel results on homomorphic images of direct products of algebras, see [4], [5].

I am also grateful to Christian Jensen for a useful pointer to the literature.

2. Nilpotence.

The concept of nilpotence of a nonassociative algebra A can be characterized in several slightly different ways. In what follows, whenever B and C are k-submodules of A, we understand BC to mean the k-submodule of A spanned by all products bc $(b\in B,\ c\in C)$. Let us define recursively k-submodules $A_{[n]}$ and $A_{(n)}$ $(n=1,2,\dots)$ of any algebra A by

(6) $$A_{[1]} \ =\ A, \qquad A_{[n+1]} \ =\ A\,A_{[n]} + A_{[n]}\,A,$$

(7) $$A_{(1)} \ =\ A, \qquad A_{(n+1)} \ =\ \textstyle\sum_{0<m<n+1} A_{(m)}\,A_{(n+1-m)}.$$

It is easy to see by induction that these yield descending chains of submodules:

(8) $$\text{for } n>0, \quad A_{[n]} \ \supseteq\ A_{[n+1]} \quad \text{and} \quad A_{(n)} \ \supseteq\ A_{(n+1)},$$

and also that

(9) $$\text{for all } n, \quad A_{[n]} \ \subseteq\ A_{(n)}.$$

If A is associative, then $A_{[n]}$ and $A_{(n)}$ clearly coincide, their common value being the submodule of A spanned by all n-fold products, which we shall write A^n. In particular for any algebra A we can apply this notation to the associative algebra $M(A)\subseteq \mathrm{Endo}(A)$, defined in the preceding section. We then have

Lemma 1. *If A is an algebra, then the following conditions are equivalent:*

(i) *There exists a positive integer n_1 such that $A_{[n_1]}=\{0\}$.*

(ii) *There exists a positive integer n_2 such that $A_{(n_2)}=\{0\}$.*

(iii) *There exists a positive integer n_3 such that $M(A)^{n_3}=\{0\}$.*

Moreover, if the above equivalent conditions hold, then letting $N_1,\ N_2,\ N_3$ be the smallest values of $n_1,\ n_2,\ n_3$ for which the indicated equations are satisfied, we have

(10) $$N_3 \ =\ \max(1,\, N_1-1), \qquad N_1 \ \leq\ N_2 \ \leq\ 2^{N_1-2}+1.$$

Proof. We first establish the stated relations between conditions (i) and (iii), and between N_1 and N_3. Let $l_A\subseteq M(A)$ denote the k-submodule of all left-multiplication operators l_x $(x\in A)$, and r_A the k-submodule of all right-multiplication operators r_x. We claim that

(11) $$\text{for all } n>0, \quad M(A)^{n+1} \ =\ (l_A+r_A)\,M(A)^n.$$

Here "$\supseteq$" is clear. To see "$\subseteq$", note that $M(A)$ consists of sums of products of one or more elements of l_A+r_A, hence $M(A)^{n+1}$ consists of sums of products of $n+1$ or more such elements. If such a product has more than $n+1$ such factors, we can, in view of associativity, group them into $n+1$ subproducts, of which the first is a single factor. (The assumption $n>0$ assures us that the first of the $n+1$ factors is not the only one.) So written, our product clearly belongs to $(l_A+r_A)M(A)^n$, proving (11).

Now (6) says that $A_{[n+1]}=(l_A+r_A)A_{[n]}$, so using (11), one concludes that

(12) $$\text{for all } n>0, \quad A_{[n+1]} \ =\ M(A)^n(A)$$

by induction from the case $n=1$. This gives the equivalence of (i) and (iii) on the one hand, and the initial equality of (10) on the other.

Turning to the submodules $A_{(n)}$, the inclusion (9) yields the implication (ii) $\Longrightarrow$ (i) and the first inequality of (10). To get the reverse implication and the final inequality of (10), we first note that these hold trivially if $A=\{0\}$, in which case $N_1=N_2=1$. To prove them for nonzero A, in which case any n_1 as in (i), or n_2 as in (ii), must be ≥ 2, it suffices to show that

(13) $$\text{for } n\geq 2, \quad A_{(2^{n-2}+1)} \ \subseteq\ A_{[n]}.$$

For $n=2$, we have equality. Assuming we know (13) for some $n\geq 2$, we look at the definition of $A_{(2^{(n+1)-2}+1)}=A_{(2^{n-1}+1)}$ as in (7), and note that in each of the summands $A_{(m)}A_{((2^{n-1}+1)-m)}$, one of the indices m or $(2^{n-1}+1)-m$ will be $\geq 2^{n-2}+1$, while the other will be at least 1; hence the summand will be contained in $A_{(2^{n-2}+1)}A + A\,A_{(2^{n-2}+1)}$. By inductive hypothesis, this is $\subseteq A_{[n]}A + A\,A_{[n]} = A_{[n+1]}$, as required. □

(If we think of an arbitrarily parenthesized nonassociative product as representing a binary tree of multiplications, the last part of the above proof is essentially a calculation showing that a binary tree with $2^{n-2}+1$ leaves $(n \geq 2)$ must contain a chain with n nodes.)

We now make

Definition 2. *An algebra A satisfying the equivalent conditions of Lemma 1 will be called* nilpotent*.*

Then we have

Corollary 3. *For any algebra A, $M(A)$ is nilpotent if and only if A is nilpotent.* □

(In the sketch in the preceding section, we defined nilpotence in terms of condition (ii) of Lemma 1, as is often done. The verification that this is equivalent to (iii) required the "$2^{n-2}+1$" part of the proof of that lemma, which is why we referred to it as not quite trivial.)

The remaining observations in this section are not needed for our main result.

In the inequality $N_1 \leq N_2$ of (10), we have equality whenever A is associative by the remark following (9). To show that the upper bound $N_2 \leq 2^{N_1-2}+1$ can also be achieved (by nonassociative algebras) for arbitrarily large N_1, take any positive integer n, and consider the k-algebra A such that

(14) A is free as a k-module on a basis $\{x_1, \ldots, x_{n-1}\}$, with multiplication given by $x_m\,x_m = x_{m+1}$ for $1 \leq m \leq n-2$, and all other products of basis elements equal to zero (including $x_{n-1}\,x_{n-1}$).

It is easy to verify by induction that for every $i \leq n$, $A_{[i]}$ is the submodule spanned by $\{x_i, \ldots, x_{n-1}\}$. In particular, $A_{[i]}$ becomes $\{0\}$ starting with $i = n$, so the N_1 of Lemma 1 is n for this algebra.

Less obvious, but no harder to verify, is the statement that

(15) for every $i \leq n$, and every j with $2^{i-2} < j \leq 2^{i-1}$, $A_{(j)}$ is also the submodule spanned by $\{x_i, \ldots, x_{n-1}\}$.

The key observation is that if $i > 1$, and j lies in the above range, then j can be written as the sum of two integers $\leq 2^{i-2}$, but not as the sum of two integers $\leq 2^{i-3}$. Using this fact, together with (7) and the definition (14), one gets (15) by induction on i.

So for this algebra, $N_2 = 2^{n-2}+1 = 2^{N_1-2}+1$, as desired.

The next lemma shows that Lie algebras behave like associative algebras in this respect.

Lemma 4. (i) *If A is an associative or Lie algebra, then for all positive integers p and q, $A_{[p]}A_{[q]} \subseteq A_{[p+q]}$.*

(ii) *If A is any algebra for which the conclusion of* (i) *holds, then for every positive integer n, $A_{[n]} = A_{(n)}$.*

Proof. For associative algebras, (i) follows from the familiar identity $A^pA^q = A^{p+q}$. For Lie algebras, we note that for $p = 1$ and q arbitrary, (i) follows from the definition of $A_{[1+q]}$; so let $p > 1$ and assume inductively that the result is true for all smaller p. Let us switch here to Lie bracket notation. By anticommutativity, the definition of $A_{[p]}$ becomes $A_{[p]} = [A,\, A_{[p-1]}]$. Using this and the Jacobi identity, we get

$$(16)\qquad \begin{aligned}[A_{[p]},\, A_{[q]}] \;&=\; [\,[A,\, A_{[p-1]}],\, A_{[q]}] \;\subseteq\; [A,\, [A_{[p-1]},\, A_{[q]}]\,] \;+\; [A_{[p-1]},\, [A,\, A_{[q]}]\,] \\ &\subseteq\; [A,\, A_{[p+q-1]}] \;+\; [A_{[p-1]},\, A_{[q+1]}] \;\subseteq\; A_{[p+q]} \;+\; A_{[p+q]} \;=\; A_{[p+q]},\end{aligned}$$

where the last two inclusions use the inductive hypothesis.

To get (ii), recall that as noted in (9), $A_{[n]} \subseteq A_{(n)}$ for arbitrary algebras; and by definition we have equality when $n = 1$. Thus, it suffices to prove $A_{[n]} \supseteq A_{(n)}$ when $n > 1$; here we may inductively assume this inclusion for smaller values of n. To get it for n, it suffices to show that each summand $A_{(m)}A_{(n-m)}$ in the definition of $A_{(n)}$ is contained in $A_{[n]}$. But by our inductive hypothesis, $A_{(m)}A_{(n-m)}$ is contained in $A_{[m]}A_{[n-m]}$, and by the assumed conclusion of (i), this is contained in $A_{[n]}$. □

3. Properties of $M(A)$.

As noted in §1, one defines the *multiplication algebra* $M(A)$ of any algebra A to be the (generally nonunital) subalgebra of the associative algebra $\mathrm{Endo}(A)$ generated by the left and right multiplication operators l_x and r_x, as x ranges over A.

For a general homomorphism of algebras $h : A \to B$, there is no natural way to map $M(A)$ to $M(B)$. For instance, if h is the inclusion of a subalgebra A in an associative algebra B, and a central element $x \in A$ becomes noncentral in B, then $l_x = r_x$ in $M(A)$, but the corresponding members of $M(B)$ are distinct. For surjective homomorphisms, however, this problem goes away:

Lemma 5. *If $h : A \to B$ is a surjective homomorphism of algebras, then there exists a unique homomorphisms $M(h) : M(A) \to M(B)$ such that*

(17) *for all* $x \in A,\quad M(h)(l_x) \ = \ l_{h(x)} \quad$ *and* $\quad M(h)(r_x) \ = \ r_{h(x)},$

equivalently, such that

(18) *for all* $u \in M(A)$ *and* $a \in A,\quad h(u(a)) \ = \ (M(h)(u))(h(a)).$

This map $M(h)$ is a surjective algebra homomorphism $M(A) \to M(B)$.

Proof. $\mathrm{Ker}(h)$ is an ideal of A, from which we see that it is carried into itself by every map l_x and every map r_x, hence by every element u of the algebra $M(A)$ generated by such maps. Hence if two elements $a,\ a' \in A$ differ by an element of $\mathrm{Ker}(h)$, so do $u(a)$ and $u(a')$; that is, if $h(a) = h(a')$, then $h(u(a)) = h(u(a'))$; so since $B = h(A)$, we get a well-defined linear map $M(h)(u) : B \to B$ as in (18).

It is immediate that $M(h)$ will be an algebra homomorphism, and that it will act by (17) on elements l_x and r_x. It will be surjective because it carries the generating set $\{l_x, r_x \mid x \in A\}$ of $M(A)$ to the corresponding generating set of $M(B)$. Since (17) is a special case of (18), the uniqueness of $M(h)$ subject to (17), which follows from the fact that the l_x and r_x generate $M(A)$, implies uniqueness subject to (18). □

It is also clear that for a composable pair of surjective algebra homomorphisms $h,\ g,$ we have $M(hg) = M(h)M(g)$; and that if we write id_A for the identity homomorphism $A \to A$, then $M(\mathrm{id}_A) = \mathrm{id}_{M(A)}$. Thus, M is a functor from the category whose objects are k-algebras and whose morphisms are *surjective* algebra homomorphisms to the category of associative k-algebras.

Let us examine the behavior of this construction on inverse limits. Suppose we are given an inverse system of k-algebras, i.e., an inversely directed partially ordered set I, a family of algebras $(A_i)_{i\in I}$, and a family of algebra homomorphisms $f_{ij} : A_i \to A_j \ \ (i \leq j)$ such that $f_{ii} = \mathrm{id}_{A_i}$ for $i \in I$, and $f_{jk} f_{ij} = f_{ik}$ for $i \leq j \leq k$. Recall that the *inverse limit* of this system can be constructed (or alternatively, the reader may consider it to be defined) as the subalgebra $A = \varprojlim_I A_i \subseteq \prod_I A_i$ consisting of those elements $(a_i)_{i\in I}$ such that $f_{ij}(a_i) = a_j$ for all $i \leq j$. Thus, the projection maps $p_j : A \to A_j$ carrying $(a_i)_{i\in I}$ to $a_j \in A_j$ satisfy

(19) $\quad f_{ij}\, p_i \ = \ p_j \ \ (i \leq j).$

The algebra A, with these maps, is universal for (19) (see [2, §§7.4-7.5] for motivation and details).

For a general inverse inverse system of algebras A_i, we cannot talk about applying M to the f_{ij} and p_i, since these may not be surjections. Indeed, even if all the f_{ij} are surjective, the resulting p_i may fail to be (cf. [7], [8], [19]). However, given any inverse system $(A_i)_{i\in I}$, and writing A for its inverse limit, if we replace each A_i with its subalgebra $p_i(A)$, the result will be an inverse system having the same inverse limit, but where the restricted maps f_{ij} and p_i are all surjective. (Actually, surjectivity of the p_i implies surjectivity of the f_{ij}, in view of (19).) Also, of course, if the original A_i were nilpotent, the new ones will still be. Hence in what follows, we shall often restrict attention to inverse systems of algebras in which all these maps are surjective.

Lemma 6. *Let $(A_i, f_{ij})_{i,j\in I}$ be an inverse system of k-algebras, and $A = \varprojlim A_i$ its inverse limit, with projection maps $p_i : A \to A_i$; and suppose the p_i (and hence the f_{ij}) are all surjective.*

Then $\varprojlim_I M(A_i)$ may be identified with a subalgebra of $\mathrm{Endo}(A)$ containing $M(A)$, by letting each $(u_i)_{i\in I} \in \varprojlim_I M(A_i)$ act on A by sending $(a_i)_{i\in I} \in A$ to $(u_i(a_i))_{i\in I} \in A$.

Proof. Recall that the condition for $(a_i)_{i\in I}$ to belong to $A = \varprojlim_I A_i$ is that each f_{ij} take a_i to a_j, and the condition for $(u_i)_{i\in I}$ to belong to $\varprojlim_I M(A_i)$ is that each $M(f_{ij})$ take u_i to u_j. By (18), with f_{ij} for h, the latter condition tells us that $f_{ij}(u_i(a_i)) = u_j(f_{ij}(a_i))$, and the former says that this equals $u_j(a_j)$. This shows that the I-tuple $(u_i(a_i))_{i\in I}$ again belongs to $A = \varprojlim_I A_i$. Thus, each $u \in \varprojlim_I M(A_i)$ induces a map $A \to A$ acting as described in the last phrase of the lemma.

It is routine to verify that these maps are module endomorphisms, that this action of $\varprojlim_I M(A_i)$ on A respects the multiplication of $\varprojlim_I M(A_i)$, and that it is faithful; so we get an identification of $\varprojlim_I M(A_i)$ with a subalgebra of $\mathrm{Endo}(A)$. Finally, for any $x = (x_i)_{i\in I} \in A$, one verifies that $(l_{x_i})_{i\in I}$ is an element of $\varprojlim_I M(A_i)$ that acts on A as l_x; so as a subalgebra of $\mathrm{Endo}(A)$, $\varprojlim_I M(A_i)$ contains each operator l_x, and we similarly see that it contains each r_x; hence it contains $M(A)$.

In fact, one easily verifies that each $u \in M(A)$ corresponds to the element $(M(p_i)(u))_{i\in I} \in \varprojlim_I M(A_i)$. □

In general, $\varprojlim_I M(A_i)$ will be properly larger than $M(A)$. Indeed, note that if all A_i are nilpotent, then the algebras $M(A_i)$ are nilpotent, hence are radical, hence $\varprojlim_I M(A_i)$ is likewise radical. But I claim that the example sketched in §1 shows that $M(A)$ can fail to be radical, hence in particular can fail be closed in $\varprojlim_I M(A_i)$ under quasiinverses. In that example (to be given in detail in §6) we took elements $x, z \in A$, and looked at the behavior of $1 - l_x\, r_z$ on A. Because x and z have nilpotent images in each of the associative algebras A_i, the images of $-\,l_x\, r_z$ in each $M(A_i)$ are nilpotent, and so quasiinvertible; and the system of quasiinverses of these images yields a quasiinverse of $-\,l_x\, r_z$ in $\varprojlim_I M(A_i)$. But the number of operations in $M(A_i)$ needed to get those quasiinverses grows with the order of nilpotence of the images of x and z, so that quasiinverse cannot be obtained by finitely many operations from l_x and r_z; and indeed does not lie in $M(A)$: if it did, it would have an image in $M(B)$ for each homomorphic image B of A, but the example constructs such a B on which $1 - l_x\, r_z$ is not invertible.

4. Hopfian modules, and modules of finite length.

Recall, however, that in the example just discussed, $1 - l_x\, r_z$, though not one-to-one, must be onto. A key to the proof of our main result will be to restrict attention to algebras B whose k-module structure makes this contrast impossible. Though in our abstract and preview, we only mentioned the condition of finite length as a k-module, we can carry out this step of our argument under a weaker assumption, which we now recall.

An algebraic structure is said to be *Hopfian* if it has no surjective endomorphisms other than automorphisms [9], [18].

Here are some quick examples of Hopfian modules: A vector space is Hopfian if and only if it is finite-dimensional. A Noetherian module M over any ring is Hopfian; for if $h : M \to M$ were surjective but not injective, then the chain

$$(20) \qquad \{0\} \subsetneq h^{-1}(\{0\}) \subsetneq h^{-1}(h^{-1}(\{0\})) \subsetneq \dots$$

would contradict the Noetherian condition [1, Prop. IV.5.3(i)] [9, Prop. 6(i)] [13, Prop. 1.14]. In particular, a module of finite length is Hopfian. Over a commutative ring, every finitely generated module is Hopfian [1, Prop. IV.5.3(ii)], and over a commutative integral domain k with field of fractions F, any k-submodule of a finite-dimensional F-vector-space is Hopfian (cf. [9, Prop. 11]). So, for instance, $\mathbb{Q}$ is a Hopfian $\mathbb{Z}$-module – though its homomorphic image $\mathbb{Q}/\mathbb{Z}$ is an example of a non-Hopfian module.

The next result only considers module-structures on A and B, and does not require the base-ring to be commutative. In view of our convention that k denotes a commutative ring, we shall call the base-ring in that result K. (In our application of the result, however, K will be our commutative ring k.)

Proposition 7. *Suppose A and B are right modules over an associative unital ring K, let $h : A \to B$ be a surjective homomorphism of modules, and let $\mathrm{Endo}(A; h)$ be the subring of the endomorphism ring $\mathrm{Endo}(A)$ consisting of the endomorphisms that carry $\ker(h)$ into itself (and hence induce endomorphisms of B).*

Suppose R is a radical subring of $\mathrm{Endo}(A)$, and B is Hopfian as a K-module. Then $R \cap \mathrm{Endo}(A; h)$ is also a radical ring; hence its image in $\mathrm{Endo}(B)$ is a radical subring of $\mathrm{Endo}(B)$.

Proof. To show that the ring $R \cap \mathrm{Endo}(A;h)$ is radical, it suffices to verify that it is closed under quasi-inverses in R. Let $r \in R \cap \mathrm{Endo}(A;h)$, and s be its quasiinverse in R. Thus, $1+r$ and $1+s$ are mutually inverse elements of $\mathrm{Endo}(A)$.

Since $1+r$ is invertible as an endomorphism of A, it is in particular surjective, from which it is easy to see that the endomorphism of B it induces is surjective. Since B is Hopfian, that endomorphism is also injective, and this says that back in $\mathrm{Endo}(A)$, $1+r$ carries no element from outside $\ker(h)$ into $\ker(h)$. Thus, the inverse map $1+s \in \mathrm{Endo}(A)$ carries no element of $\ker(h)$ out of $\ker(h)$, i.e., $1+s \in \mathrm{Endo}(A;h)$; hence $s \in \mathrm{Endo}(A;h)$, as required. □

Returning to modules and algebras over a commutative ring k, we now obtain, as part (iii) of the next lemma, a condition under which radicality implies nilpotence. (That part, in contrast to parts (i) and (ii), does not require B to be an algebra; though again, this will be so in our application.)

Lemma 8. (i) *In a radical associative algebra B, a finite set of nonzero elements $X = \{x_1, \dots, x_n\}$ $(n > 0)$ cannot satisfy $X \subseteq BX$.*

(ii) *A radical associative algebra B cannot contain a nonzero finitely generated idempotent subalgebra $S = S^2$.*

(iii) *If B is a k-module of finite length, then any radical subalgebra $R \subseteq \mathrm{Endo}(B)$ is nilpotent.*

Proof. (i): The relation $X \subseteq BX$ says that there exists an $n \times n$ matrix $\mathbf{b} = (b_{ij})$ over B satisfying $\mathbf{b}\,\mathbf{x} = \mathbf{x}$, where $\mathbf{x}$ is the column vector formed by the x_i. Hence, embedding B in a unital algebra $k+B$, $\mathbf{x}$ is annihilated by $1 - \mathbf{b} \in \mathrm{Mat}_n(k+B)$. But this is impossible, since the matrix ring $\mathrm{Mat}_n(B)$ is again radical [10, Theorem I.7.3, p.11], hence $1 - \mathbf{b}$ is invertible.

(ii): Suppose S is any idempotent subalgebra of B generated as a k-algebra by a finite set X. The fact that S is generated by X implies that $S = (k+S)X$; hence

$$(21) \qquad X \subseteq S = S^2 \subseteq B\,S = B(k+S)X \subseteq B\,X,$$

contradicting (i).

(iii): Taking generators $b_1, \dots, b_n$ for the k-module B, we note that any endomorphism u of B is determined by the element $(u(b_1), \dots, u(b_n))$; hence $\mathrm{Endo}(B)$ can be embedded as a k-module in a direct sum of n copies of B, hence it has finite length. Hence the chain of k-algebras $R \supseteq R^2 \supseteq \dots \supseteq R^n \supseteq \dots$ must stabilize; say $R^n = R^{n+1}$. This makes R^n an idempotent subalgebra of R, so by (ii), $R^n = \{0\}$. □

5. The main theorem.

Definition 9. *We shall call a k-algebra A* pro-nilpotent *if it can be written as an inverse limit of nilpotent k-algebras.*

Part (iii) of the next result is what we have been aiming at. The first two parts note what can be said under weaker conditions.

Theorem 10. *Let $B = h(A)$ be a surjective homomorphic image of a pro-nilpotent k-algebra A. Then*

(i) *For every $r \in M(B)$, the operator $1+r \in \mathrm{Endo}(B)$ is surjective. (More generally, for every $n > 0$ and $r \in \mathrm{Mat}_n(M(B))$, $1+r$ acts surjectively on the direct sum of n copies of B.)*

(ii) *If B is Hopfian as a k-module, $M(B)$ is contained in a Jacobson radical subalgebra of* $\mathrm{Endo}(B)$.

(iii) *If B is of finite length as a k-module, it is nilpotent as an algebra.*

Proof. Let $A = \varprojlim_I A_i$, where $(A_i, f_{ij})_{i,j \in I}$ is an inverse system of nilpotent k-algebras.

As noted earlier, if we replace each A_i by the image $p_i(A)$ therein, and restrict the f_{ij} to these subalgebras, we get a new inverse system having the same inverse limit A, and the restricted maps p_i and f_{ij} are surjective; moreover, the new A_i, being subalgebras of the given algebras, are still nilpotent. Hence without loss of generality, let us assume all the p_i and f_{ij} surjective.

By Corollary 3, the multiplication algebras $M(A_i)$ are nilpotent, hence are radical, and an inverse limit of radical rings is radical; so under the identification of Lemma 6, $\varprojlim_I M(A_i)$ is a radical subalgebra of $\mathrm{Endo}(A)$ containing $M(A) \subseteq \mathrm{Endo}(A;h)$.

With no additional assumptions, we see that the radicality of $\varprojlim_I M(A_i)$ implies that for every $u \in M(A)$, the operator $1+u$ is invertible on A, hence in particular, acts surjectively, hence that its image in $M(B)$ acts surjectively on B, giving the main statement of (i). This argument applies, more generally, to $\mathrm{Mat}_n(M(A))$ and $\mathrm{Mat}_n(M(B))$, acting on a direct sum of copies of A, respectively B, giving the parenthetical generalization.

If B is Hopfian as a k-module, then by Proposition 7, $(\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A;h)$ is a radical k-algebra. Since $M(A) \subseteq (\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A;h)$, its image $M(B) = M(h)(M(A)) \subseteq \mathrm{Endo}(B)$ is contained in the radical subalgebra $M(h)((\varprojlim_I M(A_i)) \cap \mathrm{Endo}(A;h))$, giving (ii).

Finally, if B has finite length, Lemma 8(iii) shows that the above radical subalgebra of $\mathrm{Endo}(B)$ is nilpotent, hence $M(B)$ is nilpotent, hence by Corollary 3, B is nilpotent. $\square$

6. Examples.

The first example below will be the promised case of a (non-Hopfian) homomorphic image B of a pro-nilpotent algebra containing elements x, z such that the map $1 - l_x r_z \in M(B)$ is not one-to-one.

In constructing that example and the next, we shall make use of unital free associative algebras $k\langle X\rangle$ in a finite set X of noncommuting indeterminates (e.g., $X = \{x, y, z\}$) over a field k, their completions, which are noncommuting formal power series algebras $k\langle\langle X\rangle\rangle$, and the nonunital versions of these two constructions (their "augmentation ideals", i.e., the kernels of the unital homomorphisms to k sending the indeterminates to zero), which we will denote $[k]\langle X\rangle$, respectively $[k]\langle\langle X\rangle\rangle$.

Within these algebras, we shall write $(a, b, \dots)$ for the 2-sided ideal generated by elements $a, b, \dots$; in the completed algebras, we shall also write $((a, b, \dots))$ for the closure of such an ideal in the inverse limit topology.

These examples will start by taking a set S of monomials in the given free generators, which does not contain the monomial 1, and forming the factor algebra $k\langle X\rangle/(S)$. Note that this has a k-basis consisting of all monomials not containing any subword belonging to S. We shall then form the completion $k\langle\langle X\rangle\rangle/((S))$ and take for our A the subalgebra $[k]\langle\langle X\rangle\rangle/((S))$. It is not hard to see that $k\langle\langle X\rangle\rangle/((S))$ is the inverse limit of the factor-algebras $k\langle X\rangle/(S \cup X^i)$ where X^i denotes the set of monomials of length i in the given generators, so that $[k]\langle\langle X\rangle\rangle/((S))$ is the inverse limit of the nilpotent algebras $A_i = [k]\langle X\rangle/(S \cup X^i)$. This inverse limit consists of all formal infinite k-linear combinations of monomials having no subword in S.

By abuse of notation, we shall use the same symbols X and $x, \dots$ for our generating set and its members in the above algebras, and for their images in our various factor-algebras.

Example 11. *There exists a pro-nilpotent associative algebra A over a field k having nonzero elements x, y, z such that $y \notin (y - xyz)$.*

Thus, the algebra $B = A/(y - xyz)$ will have the property that the operator $1 - l_x r_z$ annihilates the nonzero *element y. In particular, since $0 \neq y \in B\,y\,B$, the algebra B cannot be residually nilpotent.*

Construction and proof. Because it is easier to study the ideal of an algebra $[k]\langle\langle X\rangle\rangle/((S))$ generated by one of the indeterminates than the ideal generated by a more complicated element, we shall take for our A an algebra of the form $[k]\langle\langle x, w, z\rangle\rangle/((S))$, and find a $y \in A$ such that $w = y - xyz$.

Let us choose the set S so that the only nonzero monomials in $[k]\langle x, w, z\rangle/(S)$ are the words

(22) $\quad x^i w\, z^j \quad (i, j \geq 0)$, and subwords of such words.

Thus, we take

(23) $$S \;=\; \{xz,\, wx,\, ww,\, zw,\, zx\}.$$

Let

(24) $$A \;=\; [k]\langle\langle x, w, z\rangle\rangle/((S)).$$

For convenient calculation with ideals, we also introduce the notation

(25) $$k + A \;=\; k\langle\langle x, w, z\rangle\rangle/((S)).$$

Returning to A, which by our preceding discussion is pro-nilpotent, consider the operator $-l_x r_z \in M(A) \subseteq \varprojlim M(A_i)$. Since the latter algebra is Jacobson radical, $-l_x r_z$ is quasiinvertible in $\mathrm{Endo}(A)$; so let $y = (1 - l_x r_z)^{-1}(w)$. Clearly, this has the form

(26) $$y \;=\; w + xwz + x^2wz^2 + \dots + x^n wz^n + \dots\ .$$

(One can verify directly that this satisfies $w = y - xyz$.)

We claim that $y \notin (w)$. To see this, note that every element of (w) will be a *finite* sum

(27) $\quad \sum_{i=1}^{n} a_i\, w\, b_i \quad (a_1, \dots, a_n,\, b_1, \dots, b_n \in k + A).$

Now all monomials with factors w or z annihilate w on the left, and all monomials with factors w or x annihilate w on the right (see (22), (23)); so let us write each a_i as $a'_i + a''_i$, where $a'_i \in k[[x]]$, and the monomials occurring in a'' all have factors w or z, and each b_i as $b'_i + b''_i$, where $b'_i \in k[[z]]$ and the monomials occurring in b'' all have factors w or x. Then (27) becomes

(28) $\quad \sum_{i=1}^{n} a'_i\, w\, b'_i \quad (a'_1, \dots, a'_n \in k[[x]],\ b'_1, \dots, b'_n \in k[[z]]).$

We now see that if in (28) we take the right coefficient, in $k[[z]]$, of any term $x^j w$, this will be a k-linear combination of $b'_1, \dots, b'_n$. In particular,

(29) the k-vector-subspace of $k[[z]]$ spanned by the right coefficients in that algebra of the words $x^j w$ $(j = 0, 1, \dots)$ is finite-dimensional over k.

However, we see from (26) that in our element y, the right coefficient of $x^j w$ is z^j. The space spanned by all these elements is infinite-dimensional, so $y \notin (w) = (y - xyz)$, as claimed.

The next assertion of the example is immediate. Finally, the relations $y \in ByB \subseteq B(ByB)B \subseteq \dots$ show (using the associativity of B) that y maps to 0 in every nilpotent homomorphic image of B, so B is not residually nilpotent. □

Note that the first part of the above result implies that in $[k]\langle\langle x, y, z\rangle\rangle$ itself, the ideal generated by $y - xyz$ does not contain y; hence $[k]\langle\langle x, y, z\rangle\rangle/(y - xyz)$ gives an example with essentially the same properties. Dividing out by $((S))$ mainly made it easier for us to see what we were doing.

We can get a similar result for Lie algebras, using a Lie algebra arising from a pro-nilpotent associative algebra under commutator brackets. The construction is formally a little simpler than the above, but the verification is a little more complicated.

Example 12. *There exists a pro-nilpotent associative algebra A over a field k having nonzero elements x, y such that $y \notin (y - xy + yx)$. Thus, under commutator brackets, A is a pro-nilpotent Lie algebra with nonzero x, y such that $y \notin (y - [x, y])$ (where in this second expression, $(\ )$ is interpreted as "Lie ideal generated by").*

Hence the Lie algebra $B = A/(y - [x, y])$ has the property that $1 - \mathrm{ad}_x$ annihilates the nonzero element y. Thus $0 \neq y = [x, y] \in [B, y]$, so B is not residually nilpotent.

Construction and proof. This time, we will work with the associative algebra $[k]\langle\langle x, w\rangle\rangle/((S))$, where S is chosen so that the only nonzero monomials are the words

(30) $\quad x^i w\, x^j \ \ (i, j \geq 0)$ and their subwords.

Thus, we take

(31) $\quad S \ = \ \{w x^i w \mid i \geq 0\},$

and let

(32) $\quad A \ = \ [k]\langle\langle x, w\rangle\rangle/((S)).$

We now define

(33) $\quad y \ = \ (1 - l_x + r_x)^{-1}(w).$

Though the obvious way to begin the evaluation of this element would be to write $(1 - l_x + r_x)^{-1} = \sum_{i=0}^{\infty} (l_x - r_x)^i$, we can get the coefficient of $x^i w$ in (33) more quickly if we instead use the formula

(34) $\quad (1 - l_x + r_x)^{-1} \ = \ \sum_{i=0}^{\infty} l_x^i\, (1 + r_x)^{-1-i},$

which is valid because l_x and $1 + r_x$ commute. This gives

(35) $\quad y \ = \ \sum_{i=0}^{\infty} x^i\, w\, (1 + x)^{-1-i}.$

Again, if this lay in (w), it would follow that the right factors $(1+x)^{-1-i}$ $(i = 0, 1, \dots)$ of the monomials $x^i\, y$ would lie in a finite-dimensional k-subspace of $[k][[x]]$. But this is not so: since the positive and negative powers of $1+x$ are linearly independent in the field $k(x)$, they are linearly independent in the larger formal Laurent series field $k((x))$, hence in the smaller formal power series algebra $k[[x]] \subseteq A$.

Hence $y \notin (w) = (1-l_x+r_x)(y) = (y-xy+yx)$, and since the Lie ideal generated by $y-xy+yx = y-[x,y]$ is contained in the associative ideal generated by that element, we likewise have $y \notin (y-[x,y])$.

Again, the assertions of the final sentence follow. $\square$

In the above two examples, we took a pro-nilpotent algebra A, and were able to arrange for an element $y \in A$ to "survive" under a homomorphism $A \to B$ that made it fall together with a member of AyA or $Ay + yA$. In these cases, y survived "with the help of" other elements x and z, which did not themselves fall together with higher-degree expressions. It is natural to ask whether a family of elements can all "help one another" to survive under a homomorphism that makes each fall together with a linear combination of higher degree terms in that set of elements. One way of posing this question is: Can a homomorphic image B of a pro-nilpotent algebra A contain a nonzero subalgebra S that is idempotent, i.e., satisfies $S = S^2$?

If our algebras are associative, and the family X generating S is finite, the answer is negative. Indeed, if the pro-nilpotent algebra A is associative, then it, and hence B, will be Jacobson radical, and Lemma 8(ii) says that such an algebra cannot have a nonzero finitely generated nilpotent subalgebra. Lemma 8(i) describes a more general situation that is also excluded.

However, both of these results fail for *nonassociative* pro-nilpotent algebras. Indeed, in Example 12 we had $y \in [B, y]$, contradicting the conclusion of Lemma 8(i). We note below a simpler example with the same property (which we will want to call on for another property later), then a counterexample to the conclusion of Lemma 8(ii).

Example 13. *There exists a pro-nilpotent nonassociative algebra A over a field k having nonzero elements x, y such that $y \notin (y - xy)$.*

Hence the algebra $B = A/(y-xy)$ has the property that $1-l_x$ annihilates the nonzero element y; hence (in contrast to Lemma 8(i)) $y \in B\,y$.

Construction and proof. Paralleling our earlier constructions, the idea is to start with a nonassociative k-algebra on generators x, w, in which all monomials other than x, and w, xw, $x(xw)$, $x(x(xw)), \dots$ are set to zero. But rather than dealing with a free nonassociative algebra, and expressions with many parentheses, let us simply describe a basis for our algebra, and say how the multiplication acts on that basis. (The main value of the "free algebra modulo monomials" approach of our previous examples was to insure that the algebra constructed was associative, and no such condition is needed here.)

So let us start with an algebra having a basis $\{x, w_0, w_1, w_2, \dots\}$, and multiplication given by

(36) $\quad x\,w_i \ = \ w_{i+1} \ \ (i = 0, 1, \dots),$ and all other products of basis elements zero.

Clearly, for each $i \geq 0$, this algebra has a homomorphic image A_i in which all w_j with $j \geq i$ are set to zero, and these images form an inverse system of nilpotent algebras, whose inverse limit A consists of all formal infinite sums $\alpha x + \sum_{i=0}^{\infty} \beta_i\, w_i \ \ (\alpha,\, \beta_i \in k)$.

The ideal (w_0) of A is easily shown to consist of the finite sums $\beta_0\, w_0 + \dots + \beta_n\, w_n$. In particular, it does not contain the element $y = \sum_{i=0}^{\infty} w_i = (1-l_x)^{-1} w_0$, which satisfies $y - xy = w_0$. This gives the first assertion. The remaining assertions follow immediately. $\square$

Still more striking is

Example 14. *There exists a pro-nilpotent nonassociative algebra A over a field k having a nonzero element y such that $y \notin (y - y^2)$.*

Hence in $B = A/(y-y^2)$, the element y spans an idempotent 1-dimensional (associative!) subalgebra (in contrast to Lemma 8(ii)).

Construction and proof. This time, we start with an algebra having basis $\{w_0, w_1, \dots\}$, and multiplication given by

(37) $\quad w_i\, w_i \ = \ w_{i+1} \ \ (i = 0, 1, \dots),$ and all other products of basis elements zero.

We again get nilpotent homomorphic images A_i on setting w_j equal to zero for all $j \geq i$. The inverse limit A of these algebras consists of all formal infinite sums

(38) $\quad \sum_{i=0}^{n} \alpha_i\, w_i \quad (\alpha_i \in k).$

This time, it is not hard to see that

(39) the ideal (w_0) of A consists of all finite sums $\alpha_0\, w_0 + \dots + \alpha_n\, w_n$.

Again let $y = \sum_{i=0}^{\infty} w_i$. We find that $y - y^2 = w_0$, so $(y - y^2) = (w_0)$, the ideal described in (39), which does not contain y. This proves the main assertion, and the final statement again follows. □

Returning to associative algebras, Lemma 8(ii) tells us that in a homomorphic image B of a pro-nilpotent associative algebra A, we cannot represent each member of a finite set X of nonzero elements as a "noncommutative polynomial without linear terms" in the elements of X. But can we, perhaps, get an example in which each member of X is a "noncommuting *formal power series* without linear terms" in X? Here we understand X to be the image of a finite subset $X \subseteq A$, and the "formal power series" to be the images of such series evaluated in the pro-nilpotent algebra A.

If k is a field, the answer is still no. To see this, let us replace each of the A_i by its subalgebra generated by the image of X, then replace A by the inverse limit of these subalgebras, and B by the image of this new A. Thus, we are reduced to the case where the image of the finite set X generates each A_i. Consider now the homomorphism

(40) $\quad [k]\langle\langle X\rangle\rangle \;\to\; A$

induced by the surjective maps

(41) $\quad [k]\langle X\rangle/(X^{n(i)}) \;\to\; A_i \quad (i \in I),$

where $n(i)$ is the order of nilpotence of A_i. Inverse limits preserve surjectivity of maps between inverse systems of *finite-dimensional* vector spaces (see [12, exact sequence (2) on p.3, and Théorème 7.1 on p.57]); hence (40) is surjective. Now in $[k]\langle\langle X\rangle\rangle$, the set of elements given by formal power series in X without linear terms can be described as the left ideal $[k]\langle\langle X\rangle\rangle\, X$, so our "formal power series" assumption on B becomes

(42) $\quad X \;\subseteq\; BX,$

again contradicting Lemma 8(i).

It is not clear what we can say if k is not a field. In that case, the step used above, that an inverse limit of algebras generated by X will be generated by X as a left ideal, need not hold, as shown by

Example 15. *If k is a non-complete discrete valuation ring, then there exists a commutative associative k-algebra A containing an element x such that A is the inverse limit of a system of nilpotent k-algebras A_i each generated by the image of x, but such that the left ideal $(k + A)x$ of A generated by x is not all of A.*

Construction and proof. Let (p) be the maximal ideal of k, and $\hat{k}$ the completion of k. For each natural number i, let A_i be the nilpotent algebra $[k]\langle x\rangle/(p^i\, x^2,\, x^3)$, which we may write as $k\,x + (k/p^i\, k)\, x^2$, and which is clearly nilpotent. The inverse limit A then has the form $k\,x + \hat{k}\,x^2$, and we see that $(k + A)x = k\,x + k\,x^2 \neq A$. □

If we allow non-finitely-generated ideals and subalgebras, then the conclusions of Lemma 8(i)-(ii) also fail for associative algebras over a field, as shown by the following example of Sąsiada and Cohn. (For parallelism with the other examples of this section, we interchange below the use of the symbols x and y made in [15].) Note that the ideal (y) in the statement, though generated by a single element as a 2-sided ideal, may (and must, by Lemma 8) be infinitely generated as a left ideal and as a subalgebra.

Example 16 (Sąsiada and Cohn [15])**.** *For k a field, there exists a pro-nilpotent associative algebra A with elements x and y such that $y \notin (y - x\,y^2\,x)$.*

Thus, in $B = A/(y - x\,y^2\,x)$, the ideal (y) satisfies $B(y) = (y)$ and, as a subalgebra, $(y)^2 = (y)$. Moreover, if U is a maximal ideal of B not containing y, then in $B' = B/U$, the subalgebra (y) is simple. (*Thus, the ideal (y) of B' is a* simple *Jacobson radical algebra.*)

Construction and sketch of proof. The algebra A is $[k]\langle\langle x,\, y\rangle\rangle$. The proof that $y \notin (y - x\,y^2\,x)$ occupies most of the five pages of [15], and we will not discuss it here. (Our trick of modding out by all but a small family of monomials does not seem applicable to this case.)

In B, where the relation $y = x\,y^2\,x$ has been imposed, the asserted equalities $(y)^2 = (y)$ and $B(y) = (y)$ clearly follow.

To see the simplicity of (y) in $B' = B/U$, note first that by maximality of U, the subalgebra $(y) \subseteq B'$ contains no proper nonzero B'-ideal. Suppose, however, that in contradiction to our desired conclusion, it contained a proper nonzero (y)-ideal V. If $(y)V = \{0\}$, then the right annihilator of (y) in (y), which is clearly an ideal of B' properly contained in (y), is nonzero, a contradiction. So $(y)V \neq \{0\}$. Knowing this, we see in turn that if $(y)V(y) = \{0\}$, then the left annihilator of (y) in (y) gives the same contradiction. Hence $(y)V(y) \neq \{0\}$. But this is also clearly an ideal of B' contained in V, hence properly contained in (y), a final contradiction that completes the proof.

In the final parenthetical statement (which was the goal of [15]), radicality holds because any ideal of a radical ring is radical. □

Examples 11-14 above all show that the conclusion of Theorem 10(ii) can fail if one deletes the hypothesis that the underlying k-module of B be Hopfian. We end this section with an easy example showing that the condition that that module be Hopfian is likewise not enough to give the assertion of part (iii).

Example 17. *There exists a pro-nilpotent commutative associative k-algebra A which is Hopfian as a k-module, but not nilpotent as an algebra.*

Construction and proof. Let k be a complete discrete valuation ring, with maximal ideal (p), and consider the inverse system of nilpotent algebras $A_i = (p)/(p^i)$ $(i \geq 1)$ with the obvious connecting homomorphisms. Because k is complete, the inverse limit A of this system is isomorphic to the ideal $(p) \subseteq k$, which is free of rank 1 as a k-module, hence is Hopfian, but is not nilpotent. □

(If we had left out the assumption that k was complete, our A would have been the maximal ideal $p\,\hat{k}$ of the completion $\hat{k}$ of k. In this situation, $\hat{k}$, and hence this ideal, is again Hopfian as a k-module, but for less obvious reasons.)

7. A chain of conditions.

Theorem 10 involves a chain of successively weaker conditions on an algebra A :

(43) A is nilpotent; equivalently, $M(A)$ is nilpotent.

$\Downarrow$

(44) $M(A)$ is Jacobson radical; equivalently, for every $u \in M(A)$, $1+u$ is invertible in $k + M(A)$.

$\Downarrow$

(45) $M(A)$ is contained in a Jacobson radical subalgebra of $\mathrm{Endo}(A)$.

$\Downarrow$

(46) For every $n > 0$ and $u \in \mathrm{Mat}_n(M(A))$, $1+u$ is surjective as a map on the direct sum of n copies of A.

$\Downarrow$

(47) For every $u \in M(A)$, $1+u$ is surjective as a map $A \to A$.

We recall some quick examples showing that the first four of these conditions are distinct:

In any commutative local integral domain which is not a field, the maximal ideal A is a radical subalgebra, and satisfies $M(A) \cong A$, hence A satisfies (44), but not (43).

Amplifying the comment preceding Example 17, we note that for the algebras A of Examples 11-14 the inclusion $M(A) \subseteq \varprojlim_I M(A_i)$ yields (45), but that these algebras cannot satisfy (44), since we have seen that they have homomorphic images B on which certain operators $1+u$ $(u \in M(B))$ are non-invertible.

The algebras B of those same examples satisfy (46) and (47) by Theorem 10(i), but as we saw, they do not satisfy (45).

On the other hand, I do not know whether (46) is strictly stronger than (47).

We also note that conditions (43), (44), (46) and (47) clearly carry over to homomorphic images; but Examples 11-14 show that (45) does not; though from Proposition 7, it follows that it does when the image algebra is Hopfian as a k-module.

Condition (43) carries over to subalgebras, but none of the others do. E.g., in a discrete valuation ring, such as the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at a prime p (notation unrelated to the $A_{(n)}$ of §2!), or a formal power series algebra $k[[t]]$ over a field k, the maximal ideal (in these cases, $p\,\mathbb{Z}_{(p)}$, respectively $[k][[t]]$), regarded as an algebra, satisfies (44), and hence (45)-(47); but in these two examples, the $\mathbb{Z}$-subalgebra $p\,\mathbb{Z} \subseteq p\,\mathbb{Z}_{(p)}$, respectively the $k[t]$-subalgebra $[k][t] \subseteq [k][[t]]$, fail to satisfy (47), hence likewise (46), (45) and (44).

In the proof of Theorem 10, we obtained statement (iii) from statement (ii) essentially by showing that for an algebra of finite length as a module, condition (45) implies (43). On the other hand, we did not obtain (ii) directly from (i) – I do not know whether for algebras that are Hopfian as modules, (46) implies (45).

However, the next lemma shows that we could, alternatively, have proved (iii) directly from (i).

Lemma 18. *For A an algebra of finite length as a k-module,* (46) *implies* (43).
(*Thus, for such A,* (43)-(46) *are equivalent.*)

Proof. We shall prove the result in contrapositive form, showing, under the finite-length hypothesis, that if (43) does not hold, then (46) does not.

If (43) fails, then the decreasing chain of submodules $M(A)^n(A)$ of A never becomes zero; so by the finite length assumption, it reaches some least value. Thus, say $C = M(A)^n(A) \neq \{0\}$ satisfies

(48) $\quad C \;=\; M(A)(C).$

Being a submodule of A, C has finite length, hence it is finitely generated, say by $x_1, \dots, x_m$. The next step is just like the proof of Lemma 8(i): We form the vector $\mathbf{x} = (x_1, \dots, x_m)$, write (48) as a matrix equation $\mathbf{a}\,\mathbf{x} = \mathbf{x}$ $(\mathbf{a} \in \mathrm{Mat}_m(M(A)))$, and note that this implies that $1 - \mathbf{a} \in \mathrm{Mat}_m(\mathrm{Endo}(A))$ is not one-to-one in its action on the direct sum of m copies of A. But since that module has finite length, if $1-\mathbf{a}$ is not one-to-one, it cannot be surjective, showing that (46) fails. □

Some of conditions (44)-(47) might have further uses in the study of associative and nonassociative algebras.

8. Solvable Lie algebras.

Recall that the *derived series* of an algebra A is the sequence of subalgebras $A^{(n)}$ $(n = 0, 1, \dots)$ defined by

(49) $\quad A^{(0)} = A, \qquad A^{(n+1)} = A^{(n)}\, A^{(n)}.$

This concept is standard in the theory of Lie algebras (where the $A^{(n)}$ are in fact ideals); less so for general nonassociative algebras, though it is defined in that context in [16, p.17].

An algebra A is called *solvable* if $A^{(n)} = \{0\}$ for some $n \geq 0$. It is easy to see that $A^{(n)} \subseteq A_{(2^n)}$ $(n = 0, 1, \dots)$ so that every nilpotent algebra is solvable; but the converse is not true, as shown by the 2-dimensional Lie algebra with basis $\{x,\, y\}$ and multiplication satisfying $[x,\, y] = y$.

There is a special result on finite-dimensional Lie algebras A over a field of characteristic 0 : such an algebra is solvable if and only if its commutator ideal $A^{(1)} = [A,\, A]$ is nilpotent [11, Corollary 1 to Theorem 13, p.51]. Nazih Nahlus (personal communication) has pointed out that this fact and our main theorem have the following consequence.

Corollary 19 (to Theorem 10(iii). N. Nahlus)**.** *Let A be an inverse limit of finite-dimensional solvable Lie algebras A_i over a field k of characteristic* 0. *Then any finite-dimensional homomorphic image B of A is also solvable.*

Proof. Writing A as an inverse limit of finite-dimensional solvable Lie algebras A_i, the result quoted shows that the commutator ideals of the A_i form an inverse system of nilpotent algebras. The inverse limit $A^* \subseteq A$ of this system contains all brackets of elements of A, so when we map A homomorphically onto a finite-dimensional algebra B, Theorem 10(iii) tells us that the image of A^* is nilpotent. Since all brackets of elements of B lie in that image, B is solvable. □

However, there are *infinite*-dimensional solvable Lie algebras A in characteristic 0, and finite-dimensional solvable Lie algebras in positive characteristic, for which $A^{(1)}$ is not nilpotent. To get an example of the former, one may take the vector space A of operators on $\mathbb{R}[x]$ spanned by the operators X^n of multiplication by x^n $(n = 0, 1, \dots)$, together with the operator $D = d/dx$, and the composite operator XD, and verify that A is closed under commutator brackets, hence forms a Lie algebra. (This is the semidirect product of the 2-dimensional Lie algebra L spanned by $\{D,\, XD\}$, and the L-module $\mathbb{R}[x]$.) One finds that $A^{(1)} = [A, A]$ is spanned by all the above operators except XD. (In particular, $[D, XD] = D$ does appear.) This subalgebra is not nilpotent, since $[D, X^n] = nX^{n-1}$, so that there are elements which can be bracketed with D arbitrarily many times before going to zero. On the other hand, $A^{(2)} = [A^{(1)}, A^{(1)}]$ is spanned by the operators X^n only, and hence has zero bracket operation, so $A^{(3)} = \{0\}$, showing that A is solvable.

To get a finite-dimensional example in positive characteristic, let us first note a modification one can make in the above example. Consider the ring of functions $\mathbb{R}[x,\, e^x]$, and the space of operators on that ring spanned by D and XD as above, together with (rather than the operators X^n) the operators $X^n\, Y$ $(n \geq 0)$, where Y is the operator of multiplication by e^x. Again, one verifies that this is closed under commutator brackets, and so gives a Lie algebra A (the semidirect product of L as above and the L-module $\mathbb{R}[x]\, e^x$). Where in the preceding example, the infinite-dimensionality of $\{X^0,\, X^1,\, X^2, \dots\}$ was involved in establishing the non-nilpotence of $A^{(1)}$, here non-nilpotence follows from the single relation $[D,\, X^0\, Y] = X^0\, Y$. This does not allow us to cut our example down to a finite-dimensional subalgebra, because the iterated action of XD on $X^0\, Y$ brings in all the $X^n\, Y$. However, one finds that the structure constants of this Lie algebra with respect to our basis are integers, and that when one reduces them modulo a prime p, then the span of $\{X^p\, Y,\, X^{p+1} Y, \dots\}$ becomes an ideal. (The key calculation is that in the original algebra, $[D,\, X^p Y] = pX^{p-1} Y + X^p Y$, while in characteristic p, the first term of that expression vanishes.) The factor-algebra by that ideal is thus a $(p+2)$-dimensional solvable Lie algebra, but the relation $[D,\, X^0\, Y] = X^0\, Y$ still shows that $A^{(1)}$ is not nilpotent.

Further finite-dimensional examples in prime characteristic may be found in [6].

So if Corollary 19 is to be extended to positive characteristic, or to cover inverse limits of not necessarily finite-dimensional algebras, a very different proof will be needed.

One can, of course, generalize that corollary and its present proof by strengthening the hypothesis to *assume* A is an inverse limit of Lie algebras for which $A^{(1)}$ is nilpotent. Indeed, one can extend the resulting statement to arbitrary algebras, replacing solvability by the condition that the values of any specified family of algebra terms generate a nilpotent subalgebra.

9. Possible variants of our main theorem.

In this section, we look at a few ways Theorem 10 can, or might, be generalized.

We start with one that, as a generalization, proves disappointing, but which is instructive.

9.1. **General limits.** Recall that the concept of the inverse limit of an inversely directed system of algebraic structures is a case of the more general category-theoretic concept of the "limit" of a functor, other important examples of which are the fixed-point algebra of a group acting on an algebra, and the equalizer of a pair of morphisms [2, §§7.6] [14, §III.4]. If one examines the proof of Theorem 10, one sees no reason why it should not work for limits in this more general sense.

It does – but that extension gives nothing new:

Lemma 20. *For a k-algebra A, the following conditions are equivalent.*

(i) *A can be written as the limit of a system of nilpotent k-algebras indexed by a small category.*

(ii) *A is pro-nilpotent; i.e., it can be written as an inverse limit of an inversely directed system of nilpotent k-algebras.*

Sketch of proof. Clearly, (ii) $\implies$ (i).

Conversely, suppose $F : \mathbf{C} \to \mathbf{Alg}_k$ is a functor from a small category $\mathbf{C}$ to the category of not-necessarily-associative k-algebras, such that for all $X \in \mathrm{Ob}(\mathbf{C})$, $F(X)$ is nilpotent.

Let I be the partially ordered set of finite subsets of the object-set $\mathrm{Ob}(\mathbf{C})$, ordered by reverse inclusion, which is inversely directed. For each $i \in I$, let $\mathbf{C}_i$ be the full subcategory of $\mathbf{C}$ with object-set i, and let $A_i = \varprojlim\, (F \,|\, \mathbf{C}_i)$, where $F \,|\, \mathbf{C}_i$ denotes the restriction of F to $\mathbf{C}_i$.

Each A_i is a subalgebra of the finite product $\prod_{X \in i} F(X)$, and the class of nilpotent algebras is closed under taking finite products and passing to subalgebras, hence each A_i is nilpotent. Given $i \leq j \in I$, which by our ordering of I means $i \supseteq j$, we have a restriction functor $\mathbf{C}_i \to \mathbf{C}_j$, which leads to a homomorphism $A_i \to A_j$. It is straightforward to verify that $\varprojlim F = \varprojlim_I A_i$, yielding (ii). ☐

9.2. **Variant sorts of nilpotence.** Within the multiplier algebra $M(A)$ of an algebra A, we may look at the subalgebra $M_l(A)$ generated by the left multiplication operators l_x, and the subalgebra $M_r(A)$ generated by the right multiplication operators r_x.

If A is associative, these give nothing very new: $M_l(A)$ is isomorphic to the factor-algebra of A by its left annihilator ideal $\{x \in A \mid xA = \{0\}\}$, and $M_r(A)$ to the factor-algebra by the analogous right annihilator ideal; so each is nilpotent if and only if A is. If, rather, A is anticommutative (e.g., is a Lie algebra), or is commutative (e.g., is a Jordan algebra), then $M_l(A)$ and $M_r(A)$ coincide with $M(A)$.

But for a general nonassociative algebra A, these two subalgebras of $M(A)$ can look very different. For instance, for the algebra with multiplication (36), it is easy to see that $(AA)A = \{0\}$, so that $M_r(A)^2 = \{0\}$, but that $M_l(A)^n \neq \{0\}$ for all n.

The conditions $(\exists\, n)\, M_l(A)^n = \{0\}$ and $(\exists\, n)\, M_r(A)^n = \{0\}$ are known as *left* nilpotence and *right* nilpotence [17]. An algebra can be both left and right nilpotent without being nilpotent, as shown by the algebra with basis $x, w_0, w_1, \dots$, and multiplication

(50) $\quad x\, w_{2i} \ = \ w_{2i+1}, \quad w_{2i+1}\, x \ = \ w_{2i+2},$ all other products of basis elements being zero.

The development of Theorem 10 goes over, with no change, with the conditions of left nilpotence and right nilpotence in place of nilpotence! However, it is not clear to me what the "right" class of conditions, embracing all these sorts of nilpotence, would be, so I leave it to the experts in nonassociative rings to develop that observation further.

Here are a few further variant concepts of nilpotence, corresponding to still other subalgebras of $M(A)$:

Given any $\alpha, \beta \in k$, one can define a new multiplication on any k-algebra A by

(51) $\quad x * y \ = \ \alpha\, x\, y + \beta\, y\, x,$

(from which the original multiplication is recoverable by a transformation of the same form if $\alpha^2 - \beta^2$ is invertible in k). Left nilpotence of this operation is a property of A not in general equivalent to either nilpotence, left nilpotence, or right nilpotence of the original operation, but it necessarily has the same general properties.

Recall next that for any algebra A one can define the family of *associator* operations by

(52) $\quad a_{x,y}(z) \ = \ x(zy) - (xz)y \qquad (x, y, z \in A),$

look at the subalgebra $M_a(A) \subseteq M(A)$ generated by these maps, and consider algebras A for which $M_a(A)$ is nilpotent. Does the fact that the generating set of maps $\{a_{x,y} \mid x, y \in A\}$ is not a linear image of A but a bilinear image of $A \times A$ affect the usefulness of this construction? I don't know.

Finally, note that to every finite binary tree with n leaves, one can associate a way of bracketing n symbols, and hence a way of associating to every algebra A a derived n-ary operation. Various nilpotence-like conditions can be expressed conveniently in terms of this formalism. Thus, an algebra A is left nilpotent if and only if for some n, the $n+1$-ary operation induced by the length-n right-branching chain is zero on A; right nilpotent, likewise, if and only if for some n, the operation induced by the length-n left-branching chain is zero. (Here we call a tree a "chain" if after pruning all leaves, it has the form usually called a chain.) An algebra A is nilpotent if and only if for some n, the operations induced by all length-n chains are zero; equivalently, if and only if for some N the operations induced by all trees with N leaves are zero. An algebra is solvable if and only if for some n the operation induced by the full depth-n binary tree (with $2^{n+1} - 1$ nodes) is zero. One could put these conditions into a general framework, by associating conditions on algebras to filters of subsets of the set of finite binary trees.

9.3. **What about restricted Lie algebras?** Over a field k of characteristic $p > 0$, a more useful concept than that of a Lie algebra is that of a *restricted* Lie algebra or *p-Lie algebra*: a Lie algebra given with an additional operation, $x \mapsto x^{(p)}$, satisfying certain identities which, in associative k-algebras, relate the p-th power map with the k-module structure and commutator brackets. As a result of this additional operation, p-Lie algebras are not algebras in the sense of this note; but it would, of course, be of interest to know whether some versions of our results hold for these objects.

9.4. **What about groups?** The relation between nilpotence of Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, and nilpotence in the sense of group theory of the corresponding Lie groups, makes it natural to ask whether the methods and results of this note have analogs for groups G (not necessarily Lie).

In view of the way the brackets of a Lie algebra are related to the group operation, the natural analogs of the maps r_x and l_x in the above development would seem to be the commutator maps $c_x(y) = x^{-1}y^{-1}x\,y$. The analog of "quasiinvertibility" for a map $u : G \to G$ should be invertibility of the set-map $g \mapsto g\,u(g)$ of G to itself.

But it is not clear under what operations it would be natural to close the set of commutator maps to form the analog of $M(A)$, and whether this (or any method) will lead to an analog of Theorem 10.

10. Questions.

Various topics for further investigation have been noted above. Here are some more specific questions.

Regarding the chain of conditions in §7, we ask

Question 21. (i) *For A an algebra, is the implication* (46) $\Longrightarrow$ (47) *reversible? More generally, if an associative nonunital algebra R has a module A such that for each $r \in R$, the operator $1 + r$ is surjective on A, does the action of* $\mathrm{Mat}_n(R)$ *on the direct sum of n copies of A have the same property?*

(ii) *For A an algebra which is Hopfian as a k-module, is either of the the implications* (44) $\Longrightarrow$ (45) $\Longrightarrow$ (46) *reversible?*

Examples 14 and 16 show that a homomorphic image of a pro-nilpotent algebra can contain a simple subalgebra, and so, in particular, an idempotent subalgebra. This leaves open the question

Question 22. *Can a nonzero homomorphic image B of a pro-nilpotent algebra A over a field k be idempotent? Simple? If so, can this happen when our algebras are associative?*

(Of course, by Theorem 10(iii), such a B cannot be finite-dimensional, and by Lemma 8(i)-(ii), if our algebras are associative, B cannot be finitely generated.)

We have seen ways in which Lie algebras behave like associative algebras (Lemma 4), and ways in which they differ (the contrast between Example 12 and Lemma 8(i)). The next question notes some cases where it isn't clear on which side of the fence Lie algebras will fall.

Question 23. *Can a homomorphic image B of a pro-nilpotent Lie algebra have a nonzero finitely generated idempotent subalgebra?*

If so, can it have a nonzero finitely generated simple subalgebra?

If so, can such a subalgebra be finite-dimensional?

(Incidentally, one curious difference between the behaviors of Lie and associative algebras is noted in [3, Example 25.49], where it is observed that a topological Lie algebra (over a field) with a linearly compact topology need not be an inverse limit of finite-dimensional Lie algebras. The example is the Lie algebra spanned by $\mathbb{R}[x]$ and d/dx. Under the duality between vector spaces and linearly compact vector spaces, this shows that the "Fundamental theorem on coalgebras", a result on coassociative coalgebras, is not valid for co-Lie-algebras.)

The corollary about pro-solvable Lie algebras leads to

Question 24. *In Corollary 19, is it possible to remove or weaken* (i) *the condition that the A_i be finite-dimensional, or* (ii) *the condition of characteristic* 0, *or* (iii) *the condition that the algebras be Lie?*

In [4] and [5], N. Nahlus and the present author study homomorphic images of *direct product* algebras $\prod_I A_i$. The form of the results obtained there suggest some possible strengthenings of Theorem 10(iii):

Question 25. *In Theorem 10*(iii)*, if k is a field* (*or perhaps, more restrictively, an infinite field*)*, can the hypothesis that B is finite-dimensional* (*the form that the finite-length hypothesis takes for k a field*) *be weakened to countable-dimensional?* (*Cf.* [5, Theorem 8].)

Can the conclusion of Theorem 10(iii) *be strengthened to say that if we write $Z(B)$ for the ideal* $\{b \in B \mid b\,B = B\,b = \{0\}\,\}$, *then the composite map $A \to B \to B/Z(B)$ factors through one of the projections $p_i : A \to A_i$* (*equivalently, is continuous in the pro-discrete topology*)*?* (*Cf.* [4, Proposition 16].)

The need for the denominator "$Z(B)$" in the second paragraph of the above question can be seen from the case where all the A_i are zero-multiplication algebras.

References

[1] Hyman Bass, *Algebraic K-theory*, W. A. Benjamin, 1968. MR **40**#2736.

[2] George M. Bergman, *An Invitation to General Algebra and Universal Constructions,* pub. Henry Helson, Berkeley, CA, 1998. ISBN: 0-9655211-4-1. Readable online at http://math.berkeley.edu/∼gbergman/245 . MR **99h**:18001.

[3] George M. Bergman and Adam O. Hausknecht, *Cogroups and Co-rings in Categories of Associative Rings,* A. M. S. Mathematical Surveys and Monographs, v.45, 1996. MR **97k**:16001.

[4] George M. Bergman and Nazih Nahlus, *Homomorphisms on infinite direct product algebras, especially Lie algebras*, preprint, 33 pp., August 2009, readable at http://math.berkeley.edu/∼gbergman/papers .

[5] George M. Bergman and Nazih Nahlus, *Linear maps on* k^I, *and homomorphic images of infinite direct product algebras*, preprint, 12pp., August 2009, readable at http://math.berkeley.edu/∼gbergman/papers .

[6] Kevin Bowman and David A. Towers, *On almost nilpotent-by-abelian Lie algebras*, Linear Algebra Appl. **247** (1996) 159–167. MR **97j**:17009.

[7] Leon Henkin, *A problem on inverse mapping systems*, Proc. A. M. S . **1** (1950) 224–225. MR **11**,675e.

[8] Graham Higman and A. H. Stone, *On inverse systems with trivial limits*, J. London Math. Soc. **29** (1954) 233–236. MR **15**,773b.

[9] V. A. Hiremath, *Hopfian rings and Hopfian modules*, Indian J. Pure Appl. Math. **17** (1986) 895–900. MR **87i**:16031. Readable at http://www.new.dli.ernet.in/rawdataupload/upload/insa/INSA_2/20005a4f_895.pdf .

[10] Nathan Jacobson, *Structure of rings*, A.M.S. Colloquium Pubs., v. 37, 1956, revised 1964. MR **18**,373d, MR **36**#5158.

[11] Nathan Jacobson, *Lie Algebras*, Interscience Tracts in Pure and Applied Mathematics, No. 10, 1962; Dover, 1979. MR **26**#1345, MR **80k**:17001.

[12] C. U. Jensen, *Les foncteurs dérivés de* $\varprojlim$ *et leurs applications en théorie des modules*, Springer LNM, v.254, 1972. MR **53**#10874.

[13] T. Y. Lam, *Lectures on modules and rings,* Springer GTM, v.189, 1999. MR **99i**:16001.

[14] Saunders Mac Lane, *Categories for the Working Mathematician*, Springer GTM, v.5, 1971. MR **50**#7275.

[15] E. Sąsiada and P. M. Cohn, *An example of a simple radical ring*, J. Algebra **5** (1967) 373–377. MR **34**#2629.

[16] Richard D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, 1966; Dover, 1995. MR **35**#1643, MR **96j**:17001.

[17] A. M. Slin'ko, *The equivalence of certain nilpotencies of right-alternative rings* (Russian), Algebra i Logika **9** (1970) 342–348. MR **44**#272.

[18] K. Varadarajan, *Hopfian and co-Hopfian objects*, Publicacions Mathemàtiques **36** (1992) 293–317. MR **93i**:16002.

[19] William C. Waterhouse, *An empty inverse limit*, Proc. A.M.S. **36** (1972) 618. MR **46**#8158.

University of California, Berkeley, CA 94720-3840, USA
E-mail address: gbergman@math.berkeley.edu