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SOME EMBEDDING RESULTS FOR ASSOCIATIVE ALGEBRAS

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ABSTRACT. Suppose we wish to embed an (associative) k -algebra A in a k -algebra R generated in some specified way; e.g., by two elements, or by copies of given k -algebras A_1, A_2, A_3 . Several authors have obtained sufficient conditions for such embeddings to exist. We prove here some further results on this theme. In particular, we merge the ideas of existing constructions based on two generating *elements*, and on three given *subalgebras*, to get a construction using two given subalgebras.

We pose some questions on how far these results can be strengthened.

Below, rings and algebras will be associative and, except where the contrary is stated, unital, with homomorphisms respecting 1. “Countable” will mean finite or countably infinite.

I use, in several places below, techniques based on the Diamond Lemma, in particular, on Theorems 1.2 and 6.1 of [2]. I have worded the arguments where these are first used so that the reader unfamiliar with [2] can see more or less what is involved. But for precise formulations and proofs, see that paper.

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For other sorts of results on embedding general k -algebras in finitely generated ones, sometimes called “affinization”, see [1] (where the emphasis is on controlling the Gel’fand-Kirillov dimension), and works referenced there.

1. ALGEBRAS WITH FEW GENERATORS

Let me begin with a result which we shall subsequently strengthen in several ways, but which gives a simple illustration of a technique we shall frequently use.

Proposition 1. *Let k be a commutative ring, and A a countably generated k -algebra which is free as a k -module on a basis containing 1. Then A can be embedded in a k -algebra R generated by three elements.*

In fact, given any countable generating set $S = \{s_0, s_1, \dots, s_n, \dots\}$ for A as a k -algebra, one can take $R \supseteq A$ to have generators x, y, z such that

$$(1) \quad x y^n z = s_n \quad (n = 0, 1, \dots).$$

Proof. Let $\{1\} \cup B$ be a basis for A as a k -module, and assume for convenience that B does not contain any of the symbols x, y, z . In describing a presentation of R , we will want to distinguish between algebra elements and *expressions* for those elements; so for every $a \in A$, let $\varepsilon(a)$ denote the unique expression for a as a k -linear combination of elements of $\{1\} \cup B$.

We shall prove our result by applying the Diamond Lemma, Theorem 1.2 of [2], to a presentation of R , not in terms of x, y and z , but in terms of the larger generating set

$$(2) \quad \{x, y, z\} \cup B,$$

using both the relations corresponding to (1), namely

$$(3) \quad x y^n z = \varepsilon(s_n) \quad (n = 0, 1, \dots),$$

and relations which describe A by stating how members of B are multiplied,

$$(4) \quad b b' = \varepsilon(b b') \quad (b, b' \in B).$$

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We view each of the relations in (3) and (4) as a *reduction rule*, which specifies that the monomial of length ≥ 2 on its left-hand side is to be reduced to the k -linear combination of elements of $\{1\} \cup B$ on the right. Note that each of these rules carries monomials to linear combinations of shorter monomials. Hence the partial order on the free monoid on our generating set (2) that makes $s \leq t$ if and only if either $s = t$, or s is strictly shorter than t , satisfies the hypotheses of [2, Theorem 1.2]; namely, in addition to respecting the monoid structure and having descending chain condition, it has the property that the output of each of our reductions is a linear combination of monomials \leq the input monomial with respect to the ordering. None of the monomials on the left-hand sides of our relations are subwords of others, and the only monomials that can be formed by overlap of two such monomials are those of the form $bb'b''$ ($b, b', b'' \in B$), so these give the only “ambiguities” in the sense of [2, §1]. Those ambiguities are “resolvable” – i.e., the two possible reductions that can be applied to an ambiguously reducible monomial $bb'b''$, when followed by appropriate further reductions, lead to a common value – because A is associative.

Hence, by [2, Theorem 1.2], the algebra R presented by generators (2) and relations (3) and (4) has as a k -module basis the set of all monomials w in the generating set (2) such that no subword of w is the left-hand side of any of the relations of (3) or (4). In particular, $\{1\} \cup B$ is a subset of this basis, and by the relations (4), the k -submodule of R spanned by that subset is a k -subalgebra isomorphic to A . \square

It was shown in [5] that one can in fact do the same using two, rather than three generators, with the help of a slightly less obvious family of monomials.

Proposition 2 (after [5, Theorem 3.2]). *Let k , A and S be as in Proposition 1. Then A can be embedded in a k -algebra R generated by two elements, x and y , so that*

$$(5) \quad x^2 y^{n+1} x y = s_n \quad (n = 0, 1, \dots).$$

Proof. Note that the words on the left-hand side of (5) involve x^2 only in the leftmost position, and have no nonempty subword of x^2 at their right end. This limits possible overlaps or inclusions among such words to the inclusion of one as a left segment of another. But any two distinct words of that sort differ in the position of the next x after the initial x^2 , making such inclusions impossible.

The rest of the proof follows exactly the proof of Proposition 1, with $xy^n z$ everywhere replaced by $x^2 y^{n+1} x y$. \square

Alternatively, one can get the fact that a k -algebra containing A can be generated by two elements from the fact that it can be generated by three, using the lemma on p.1096 of [12], which notes that if an algebra (they say “ring”, but the argument works equally for algebras) R is generated by n elements r_1, \dots, r_n , then the $(n+2) \times (n+2)$ matrix ring $M_{n+2}(R)$ can be generated by 2 elements. (Namely, by one matrix which permutes the n coordinates cyclically, and one having first two rows $(0, r_1, \dots, r_n, 0)$ and $(1, 0, \dots, 0)$, and all other rows zero.) That lemma is used in [12] to show, as in Proposition 2 above and Theorem 4 below, that countably generated rings can be embedded in 2-generator rings. But the technique for going from countable to finite generation is quite different from that used in most of this note; we will look at it in the final section.

The proof of Proposition 2 given above is essentially the one given in [5], using a tool equivalent to the Diamond Lemma, which the authors call the method of Gröbner-Shirshov bases and apply with k assumed a field. (Cf. also [4], [6].) As a statement that any countably generated algebra is embeddable in a 2-generated algebra, the above result is attributed there to Mal’cev [10]. However, we shall see in §3 that the proof in [10] uses a different construction, which yields embeddings of nonunital, but not in general of unital algebras, and which leads to some further interesting ideas.

In the two preceding results, we applied the Diamond Lemma, but not to the generating set $\{x, y, z\}$ or $\{x, y\}$ that we might have expected to use. In the proof of the corollary below, we apply the above proposition, but not over the base ring one might expect.

Corollary 3 (cf. [12, Proposition 2]). *Let k be a commutative ring, and A a countably generated commutative k -algebra. Then there exists a (generally noncommutative) k -algebra R which is generated as a k -algebra by two elements, and contains A in its center.*

Proof. Let us apply Proposition 2 with A in the role of both the k and the A of that proposition, and with the role of S played by any countable generating set for A over k . Since A is free as an A -module on the basis $\{1\} \cup \emptyset$, the empty set will play the role of B . Proposition 2 now gives us a faithful A -algebra R generated over A by elements x and y satisfying the relations (5). Since by hypothesis the elements

on the right hand sides of these relations generate A over k , the k -subalgebra of R generated by $\{x, y\}$ contains A , hence is all of R . Since R was constructed as an A -algebra, A is central in R . \square

For instance, taking $k = \mathbb{Z}$, and for A any countable commutative ring, we get a 2-generated \mathbb{Z} -algebra which is a faithful A -algebra. Here A might be \mathbb{Q} , or an extension field of \mathbb{Q} of countable transcendence degree; or it might be a commutative ring having any countable Boolean ring as its Boolean ring of idempotents. (I gave a construction of a finitely generated algebra with an infinite Boolean ring of central idempotents, using (1), at the end of §12.2 of [3]. The present note had its origin in thinking about how that construction might be generalized.)

The contrast with commutative algebras is striking. If k is a field and R a finitely generated commutative k -algebra, I claim that any subfield A of R containing k must be a finite extension of k (i.e., finite-dimensional). For by Theorem IX.1.1 of [9], R admits a homomorphism h into the algebraic closure \bar{k} of k ; so as R is finitely generated as an algebra and \bar{k} is algebraic, $h(R)$ must be a finite extension field of k . Since A is a field, h is one-to-one on A , so A itself must be finite over k , as claimed. Likewise, since a finitely generated commutative algebra over a field is Noetherian, its Boolean ring of idempotents cannot be infinite.

It is noted in [12, proof of Corollary 2] that any finitely generated \mathbb{Z} -algebra which contains \mathbb{Q} is an example of a \mathbb{Q} -algebra which *cannot* be written $R \otimes_{\mathbb{Z}} \mathbb{Q}$ for R a \mathbb{Z} -algebra which is free as a \mathbb{Z} -module.

2. MORE GENERAL MODULE-STRUCTURES

Propositions 1 and 2, which we proved using the “everyday” version of the Diamond Lemma, require the algebra A to be free as a k -module on a basis of the form $B \cup \{1\}$. Using the bimodule version of the Diamond Lemma, we can drop that condition.

Theorem 4 (cf. [12, Theorem on p.1097]). *Let k be a commutative ring, and A any k -algebra generated as a k -algebra by a countable set $\{s_0, s_1, \dots\}$. Then A can be embedded in a k -algebra R generated by three elements x, y, z so that (1) holds, and also in a k -algebra generated by two elements x, y so that (5) holds.*

Proof. We shall prove the case based on (1). The case based on (5) is exactly analogous.

Given A and $\{s_i\}$ as above, let us use the bimodule version of the Diamond Lemma, Theorem 6.1 of [2], with our A in the role of the (not necessarily commutative) ring called k in that theorem. We begin by taking three (A, A) -bimodules freely generated by k -centralizing elements x, y and z ,

$$(6) \quad M_x = Ax A \cong A \otimes_k A, \quad M_y = Ay A \cong A \otimes_k A, \quad M_z = Az A \cong A \otimes_k A,$$

and forming the tensor ring $A\langle M_x \oplus M_y \oplus M_z \rangle$ on their direct sum. If we grade this ring in the obvious way by the free monoid on $\{x, y, z\}$, its homogeneous component indexed by each word $xy^n z$ ($n \geq 0$) is the product

$$(7) \quad \begin{aligned} M_x (M_y)^n M_z &= Ax (Ay)^n Az A \\ &\cong (Ax A) \otimes_A (Ay A) \otimes_A \cdots \otimes_A (Ay A) \otimes_A (Az A) \cong A \otimes_k A \otimes_k \cdots \otimes_k A \quad (n+3 \text{ A's}). \end{aligned}$$

We now impose on $A\langle M_x \oplus M_y \oplus M_z \rangle$ relations determined by reduction maps sending the homogeneous component indexed by each word $xy^n z$ to the component A (indexed by the empty word 1) using the (A, A) -bimodule homomorphism that acts on reducible elements of (7) by

$$(8) \quad a_0 x a_1 y a_2 \cdots a_n y a_{n+1} z a_{n+2} \longmapsto a_0 a_1 a_2 \cdots a_n a_{n+1} s_n a_{n+2}.$$

To see that such a homomorphism exists, we note first that the description of the bimodule (7) as an $n+3$ -fold tensor product over k of copies of A (last step of (7)) shows that (8) determines a k -module homomorphism, by the universal property of \otimes_k . Looking at how the right-hand side of (8) depends on a_0 and a_{n+2} , we see that this map is in fact a homomorphism of (A, A) -bimodules.

We note next that the case of (8) where $a_0 = \cdots = a_{n+2} = 1$ shows that this map carries the left hand side of (1) to the right hand thereof.

Because the family of reductions $xy^n z \mapsto 1$ in the free monoid on $\{x, y, z\}$ has no ambiguities, Theorem 6.1 of [2] shows that the A -ring R presented by the (A, A) -bimodules M_x, M_y, M_z and the relations equating inputs and outputs of each bimodule homomorphism (8) is the direct sum of all iterated tensor products of those three bimodules in which no subproduct $M_x (M_y)^n M_z$ ($n \geq 0$) occurs. In particular, the component of R indexed by the monoid element 1 is one of these summands, and is a copy of the algebra A .

A priori, the ring R we have constructed is generated by A and the three bimodules (6). But since the relations (1) and the structure of A allow us to express all elements of A in terms of x , y , z and the elements of k , and since all elements of $M_x = Ax A$, $M_y = Ay A$ and $M_z = Az A$ can then be expressed using these elements and x , y and z , we see that R is, as claimed, generated over k by those three elements, completing the proof of the case of the theorem based on generators x , y and z and relations (1).

The case based on generators x and y and relations (5) is exactly analogous. \square

Incidentally, we could have carried out the above construction equally well with the terms on the right-hand side of (8) permuted in any way which left a_0 and a_{n+2} fixed. Those two have to be placed as shown, to make the map an (A, A) -bimodule homomorphism, but the decision on how to order the others, in particular, of where to place the s_n , was quite arbitrary.

3. THE IDEAL EXTENSION PROPERTY

Let us look at Theorem 4 from a different point of view.

Definition 5. *If $A \subseteq B$ are algebras, we will say that A has the ideal extension property in B if every ideal $I \subseteq A$ is the intersection of A with an ideal $J \subseteq B$; equivalently, is the intersection of A with the ideal BIB of B that it generates.*

(This is somewhat like the *lying-over* property of commutative ring theory; but since the latter concerns prime ideals, we do not use that name, but one modeled on the *congruence extension* property of universal algebra [8, p.412].)

Now in Theorem 4, the countably generated algebra A can be thought of as the factor-algebra of the free k -algebra on a countably infinite set of generators by an *arbitrary* ideal I ; and the conclusion shows us that $k\langle x, y, z \rangle$ (respectively, $k\langle x, y \rangle$) has a homomorphic image in which the free subalgebra generated by the elements $xy^n z$ (respectively, $x^2 y^{n+1} xy$) collapses to an isomorphic copy of A . So the theorem tells us that that free subalgebra on countably many generators has the ideal extension property within the given 2- or 3-generator free algebra.

In fact, the method of proof of that theorem clearly shows the following.

Corollary 6 (to proof of Theorem 4). *Let k be a commutative ring, X a set, and W a family of nonempty words in the elements of X (i.e., elements of the free semigroup on X), such that no member of W is a subword of another, and no nonempty proper final subword of a member of W is also an initial subword of a member of W . Then the subalgebra of $k\langle X \rangle$ generated by W is free on W , and the inclusion $k\langle W \rangle \subseteq k\langle X \rangle$ has the ideal extension property.*

Proof. Like the proof of Theorem 4. (Note that the “no inclusions and no overlap” assumption on W is precisely what is needed for a reduction system mapping into A every tensor product $M_{x_1} \otimes_k \cdots \otimes_k M_{x_n}$ such that $x_1 \dots x_n \in W$ to have no ambiguities.) \square

4. SOME OLD RESULTS ON NONUNITAL EMBEDDINGS

The two earliest results I am aware of which showed that wide classes of associative algebras could be embedded in two-generator algebras, Theorem 3 of [10] and Lemma 2 of [13], were obtained by methods that, in effect, established the ideal extension property (Definition 5 above) for certain free subalgebras of free algebras, without the use of anything like the Diamond Lemma. (And which, consequently, did not yield normal forms for the 2-generator algebras R obtained.)

Those results concerned non-unital algebras, so we make

Definition 7. *In this section, k -algebras, though still associative, will not be assumed unital. (In formal statements we will make this explicit, using the word “nonunital”, meaning “not necessarily unital”. On the other hand, our commutative base ring k will continue to be unital.)*

The free non-unital k -algebra on a set X will be denoted $[k]\langle X \rangle$. For R a nonunital k -algebra, we shall write $k + R$ for the unital k -algebra obtained by universally adjoining a unit to R . Thus, $k + R$ has underlying k -module $k \oplus R$.

The ideal extension property for nonunital algebras will be defined as for unital algebras. The one formal change required is that the ideal J of B generated by $I \subseteq A$ must be described as $(k + B)I(k + B)$ rather than BIB .

Let us give a name to a property which is implicit in the arguments of [10] and [13].

Definition 8. We shall call a subsemigroup S of a semigroup T isolated if for all $t, t' \in T \cup \{1\}$ and $s \in S$, one has $tst' \in S \implies t, t' \in S \cup \{1\}$. (Here we write $-\cup\{1\}$ for the construction of adjoining 1 to a semigroup, to get a monoid.)

Lemma 9 (after [13, proof of Lemma 2]). If S is an isolated subsemigroup of a semigroup T , and k is any commutative ring, then the semigroup algebra kS has the ideal extension property in the semigroup algebra kT .

Proof. If I is an ideal of kS , then the general element of the ideal $J = (k + kT)I(k + kT)$ generated by I in kT can be written

$$(9) \quad g = \sum_{i=1}^n t_i f_i t'_i,$$

where each f_i lies in I , and all t_i and t'_i lie in $T \cup \{1\}$. Let us write $g = g' + g''$, where g' is the sum of those terms of (9) which have both t_i and t'_i in $S \cup \{1\}$, and g'' is the sum of all other terms. Then clearly $g' \in I$, while by the assumption that S is isolated in T , the element g'' is a k -linear combination of elements of $T - S$. Hence if $g \in kS$, so that $g'' = g - g' \in kS$, we must have $g'' = 0$, so $g = g' \in I$. This shows that $J \cap kS = I$, as required. \square

We also note

Lemma 10. In a free semigroup, every isolated subsemigroup is free.

Proof. Let S be an isolated subsemigroup of the free semigroup T , and W the set of elements of S that cannot be factored within S . Then every member of S can be written as a product of members of W , and it suffices to show that this factorization is unique. Suppose

$$(10) \quad u_1 \dots u_m = v_1 \dots v_n \quad (m, n \geq 2, u_i, v_j \in W),$$

and assume inductively that for every member of S of smaller length in the free generators of T , the expression as a product of members of W is unique. Without loss of generality, we may assume the length of v_1 in the free generators of T to be greater than or equal to that of u_1 , and so write $v_1 = u_1 w$ for some $w \in T \cup \{1\}$. Applying the definition of isolated subsemigroup to the equation $v_1 = 1 \cdot u_1 \cdot w$, we conclude that $w \in S \cup \{1\}$. Hence as v_1 cannot be factored in S , we must have $w = 1$, hence $v_1 = u_1$; so (10) implies $u_2 \dots u_m = v_2 \dots v_n$. By our inductive assumption, these factorizations are the same; so the two factorizations of (10) are the same. \square

Remark: If $S \subseteq T$ are monoids, then S is isolated in T if and only if it is closed under taking factors. (“If” is immediate; “only if” can be seen by applying Definition 8 with $s = 1$.) Hence the isolated submonoids of a free monoid are just the submonoids generated by subsets of the free generating set, which will be uninteresting for our purposes. But there are many interesting isolated subsemigroups of free semigroups. The next result notes a family of examples implicit in the two papers referred to.

Lemma 11. If f is a function from the positive integers to the positive integers, then in the free semigroup T on two generators x and y , the subsemigroup S generated by all elements

$$(11) \quad x y^n x^{f(n)} \quad (n \geq 1)$$

is isolated. (The case where $f(n) = n$ is used by Mal'cev [10]; the case $f(n) = 1$ by Shirshov [13].)

Proof. For S as described, it is not hard to see that we can test a word in T for membership in S , and, if it does lie in S , parse it uniquely as a product of the generators (11), as follows. We check that it begins with an x , followed by one or more occurrences of y . If the number of y 's immediately following x is n , then after these there should either be exactly $f(n)$ x 's, and the word should terminate there, or there should be $f(n) + 1$ x 's, in which case we lop off all but the last of these, and test and parse the remainder in the same way.

Given the above observation, suppose that $u = tst'$ ($t, t' \in T \cup \{1\}$, $s \in S$) lies in S . If $t \notin S \cup \{1\}$, then in the process of parsing u as above, the end of its initial string t must come after we have begun, but before we have finished, some factor $x y^n x^{f(n)}$ in the expression for u as a product of such factors. Moreover, we must have gotten more than the initial x of that factor, since that must be followed in u by a nonempty string of y 's, but s begins with an x . If we have only gotten a subword of the form $x y^m$ ($1 \leq m \leq n$), then since the next letter is an x which does not end the word u , it should be the first of $f(m) + 1 \geq 2$ x 's; but that x is followed by a y , so this cannot be the case. Finally, if we have gotten $x y^n x^r$ ($r < f(n)$), then the fact that s begins with $x y$ implies that, bringing in the lone initial x of s ,

we get a string of $r + 1 < f(n) + 1$ x 's, but again this string does not terminate our word (since $s \neq x$), another contradiction.

So, rather, we must have $t \in S \cup \{1\}$. Thus $ts \in S$, and our parsing process, when it reaches the end of ts , requires that what remains, namely t' , also belong to $S \cup \{1\}$. This completes the proof that S is isolated in T . \square

Combining the last three lemmas, we have

Theorem 12 (after Mal'cev [10, Theorem 3], Shirshov [13, Lemma 2]). *Let k be a commutative ring, let B be the free nonunital associative algebra $[k]\langle x, y \rangle$, and let A be either the subalgebra of B generated by all monomials $xy^n x$, or the subalgebra generated by all monomials $xy^n x^n$ (or, more generally, the subalgebra generated by all monomials $xy^n x^{f(n)}$ for any function f from the positive integers to the positive integers) for $n \geq 1$.*

Then B is a free algebra on the indicated countably infinite generating set, and has the ideal extension property in A .

This gives, for nonunital algebras, another way of embedding an arbitrary countably generated k -algebra in a 2-generator k -algebra. \square

We remark that Shirshov's statement of Lemma 2 of [13] leaves it unclear whether unital or nonunital algebras are intended. However, in the unital case, if we write $B = k\langle x, y \rangle$, the unital subalgebra $A \subseteq B$ generated by the elements $xy^n x$ does *not* have the ideal extension property, which his proof would require. For example, let I be the ideal of A generated by xyx and $xy^2x - 1$. Clearly I is proper, since the factor-algebra A/I is free on the free generators $xy^n x$ of A other than xyx and xy^2x . However, the ideal J that it generates in B is improper, since in B/J , the element $xyxy^2xyx$ reduces, on the one hand, to 0, in view of the factors xyx , while on the other hand, if we simplify the middle factor xy^2x to 1, and then do the same to the resulting monomial, we get 1. So $0 = 1$ in B/J , so $J = B$, so $J \cap A = A \neq I$. Thus, the algebras of [13, Lemma 2] should be understood to be nonunital.

The converse of Lemma 11 above is not true. For example, in the free semigroup T on one generator x , the subsemigroup S generated by x^2 is not isolated (since $x \cdot x^2 \cdot x \in S$), but $kS \subseteq kT$ does have the ideal extension property for every k . This suggests

Question 13. *Is there a nice characterization of the inclusions $S \subseteq T$ of semigroups (respectively, monoids) for which the inclusion of nonunital (respectively, unital) k -algebras $kS \subseteq kT$ has the ideal extension property?*

In particular, what can one say in the cases where S and T are free as semigroups or monoids?

(One expects the answers to the above questions to be independent of k , but there is no evident reason why this must be true. It is not too implausible that it might depend on the characteristic of k .)

5. EMBEDDING IN ALGEBRAS GENERATED BY A GIVEN FAMILY OF ALGEBRAS

The condition of countable generation on the algebra A in the results of the preceding sections cannot be dropped. For instance, if k is a field, every finitely generated k -algebra is countable-dimensional, hence so is every algebra embeddable in such an algebra. So, for example, a finitely generated algebra over the field \mathbb{R} of real numbers cannot contain a copy of the rational function field $\mathbb{R}(t)$, since that is continuum-dimensional.

To get around this difficulty, we might vary the construction of Proposition 1 by considering k -algebras generated by elements x and z together with all formal real powers y^r ($r \in \mathbb{R}$) of the symbol y . We would then have enough expressions $xy^r z$ to hope to get any continuum-generated k -algebra A . In effect, we would be looking at k -algebras generated by x , z and a copy of the group algebra kG , where G is the additive group of the real numbers, written multiplicatively as formal powers of y .

We can, in fact, get such results with kG replaced by a fairly general k -algebra. Here is one such statement (where " A_0 " is the algebra we want to embed, and " A_1 " the algebra generalizing kG).

Theorem 14. *Suppose A_0 and A_1 are faithful algebras over a commutative ring k , such that k is a module-theoretic direct summand in each, and such that A_0 is generated as a k -algebra by the image of a k -module homomorphism $\varphi : A_1 \rightarrow A_0$.*

Then A_0 can be embedded in a k -algebra R generated over A_1 by two elements x and z satisfying

$$(12) \quad xaz = \varphi(a) \quad (a \in A_1).$$

(Here for notational convenience we are identifying A_0 and A_1 with their embedded images in R .)

Proof. Our first step will be to embed A_0 and A_1 in a common k -algebra A having a k -module endomorphism θ that carries A_1 to our generating subset of A_0 . To do this, let us choose k -module decompositions of the sort whose existence is assumed in the hypothesis,

$$(13) \quad A_0 = k \oplus M_0, \quad A_1 = k \oplus M_1.$$

Letting $A = A_0 \otimes_k A_1$ (made a k -algebra in the usual way), we see from (13) that the k -algebra homomorphisms $a_0 \mapsto a_0 \otimes 1$ and $a_1 \mapsto 1 \otimes a_1$ are embeddings of A_0 and A_1 in A . Letting $\pi : A_0 \rightarrow k$ be the k -module projection along M_0 , we find that the k -module endomorphism θ of A given by

$$(14) \quad \theta(a_0 \otimes a_1) = \varphi(a_1) \otimes \pi(a_0)$$

carries $k \otimes_k A_1$, our copy of A_1 , onto $\varphi(A_1) \otimes_k k$, the generating k -submodule for our copy of A_0 .

Now that we have A and θ , the remainder of our proof is like that of Theorem 4, but simpler. We take two (rather than three) free k -centralizing (A, A) -bimodules,

$$(15) \quad M_x = AxA \cong A \otimes_k A, \quad M_z = AzA \cong A \otimes_k A,$$

form the tensor ring $A\langle M_x \oplus M_z \rangle$ on their direct sum, and impose the relations determined by a single bimodule homomorphism from the component indexed by xz , namely $M_x \otimes_A M_z = AxAzA$, to the component indexed by 1, namely A , where this homomorphism is defined to act on generators by

$$(16) \quad axa'za'' \mapsto a\theta(a')a''.$$

On the indexing free monoid on $\{x, z\}$, this corresponds to the single reduction $xz \mapsto 1$, which has no ambiguities. As in the proof of Theorem 4, we deduce that the relations corresponding to (16) define a k -algebra R in which A is embedded. Hence A_0 and A_1 are embedded in R , where they satisfy (12), from which the assertions of the theorem follow. \square

In fact, there is a result in the literature which achieves much greater generality in some ways (though in others it is more restricted). Bokut' shows in Theorems 1 and 1' of [4] that for any four nonzero nonunital algebras A_0, A_1, A_2, A_3 over a field k , one can embed A_0 in an algebra R generated by the union of one copy of each of A_1, A_2 and A_3 , as long as A_0 satisfies the obvious restriction of having k -dimension less than or equal to that of the k -algebra coproduct of A_1, A_2 and A_3 (namely, $\max(\aleph_0, \dim A_1, \dim A_2, \dim A_3)$), and (for less obvious reasons; but see note at reference [5] below) as long as $\text{card } k$ is less than or equal that same dimension. Moreover, Bokut's construction makes R a simple k -algebra!

So Theorem 14, in the case where k is a field, and our algebras are nonunital, and the cardinality of k satisfies the indicated bound, is majorized by the particular case of Bokut's result where A_2 and A_3 are free algebras on single generators x and z .

Given that Proposition 2 and Theorem 12 above improve on our original $xy^n z$ construction by using two generators rather than three, it is natural to ask whether one can get a result that embeds an algebra A_0 in an algebra R generated by copies of two given algebras, A_1 and A_2 , rather than the three of the result quoted. We obtain such a result, Theorem 15, below (though the algebras allowed are not quite as general as I would like; and I do not attempt to make R simple).

Let us recall, before going further, that nonunital k -algebras R correspond to unital k -algebras R' given with augmentation homomorphisms $\pi : R' \rightarrow k$, via the constructions $R' = k + R$ and $R = \ker(\pi)$, and that these constructions in fact give an equivalence between the category of nonunital k -algebras and the category of augmented unital k -algebras. From this point of view, the condition in Theorem 14 above that A_0 and A_1 each have k as a k -module direct summand is a weakened version of nonunitality, a "module-theoretic augmentation" rather than a ring-theoretic one. In the next result, we likewise have augmentation-like conditions of various strengths on the three given algebras. That is not surprising, since the result is modeled on Theorem 12.

Theorem 15. *Let k be a commutative ring, and let A_1 and A_2 be k -algebras such that*

(i) *the structure map $k \rightarrow A_1$ admits a module-theoretic left inverse π , whose kernel we shall denote M_1 , and*

(ii) *A_2 admits a surjective k -algebra homomorphism $\psi : A_2 \rightarrow k[x]/(x^3)$, whose kernel we shall denote M_2 . We shall, by abuse of notation, use the same symbol x for the image in $k[x]/(x^3)$ of $x \in k[x]$, and also for a fixed inverse image, in A_2 , of that element of $k[x]/(x^3)$ under ψ .*

Then in the coproduct ("free product")

$$(17) \quad B = A_1 \amalg A_2$$

of A_1 and A_2 as k -algebras, the k -submodule xM_1x is isomorphic to M_1 , and generates a nonunital k -subalgebra A isomorphic to the nonunital tensor algebra $[k]\langle M_1 \rangle$; and this subalgebra A has the ideal extension property in B .

Hence, any k -algebra A_0 which admits a k -algebra homomorphism to k (an augmentation), and which can be generated as a k -algebra by a module-theoretic homomorphic image of M_1 , can be embedded in a k -algebra R generated by an image of A_1 and an image of A_2 . Moreover, these images can be taken to be isomorphic copies of those two algebras.

Proof. By (i),

$$(18) \quad A_1 = k \oplus M_1$$

as k -modules, while (ii) leads to a decomposition

$$(19) \quad A_2 = k \oplus kx \oplus kx^2 \oplus M_2.$$

So writing

$$(20) \quad M'_2 = kx \oplus kx^2 \oplus M_2,$$

we have

$$(21) \quad A_2 = k \oplus M'_2.$$

By Corollary 8.1 of [2], the decompositions (18) and (21) lead to a decomposition of the k -algebra coproduct $B = A_1 \amalg A_2$ as the k -module direct sum of all alternating tensor products

$$(22) \quad \cdots \otimes_k M_1 \otimes_k M'_2 \otimes_k M_1 \otimes_k M'_2 \otimes_k \cdots,$$

(where each such tensor product may begin with either M_1 or M'_2 and end with either M_1 or M'_2 , and where we understand the unique length-0 tensor product to be k , and the two length-1 products to be M_1 and M'_2).

Using (20), we can now refine this decomposition, writing B as the direct sum of submodules each of which is

$$(23) \quad \text{a tensor product such that, as in (22), every other term is } M_1, \text{ but where each of the remaining terms can be any of the three } k\text{-modules } kx, kx^2, \text{ or } M_2.$$

Let us note that if we multiply two of the summands (23) together within B , the result will *often* lie entirely within a third. The exception is when the first factor ends with M_1 and the second begins with M_1 , in which case the relation

$$(24) \quad M_1 M_1 \subseteq k + M_1,$$

arising from the relatively weak module-theoretic hypothesis (i) on A_1 , leads to two such summands.

We now consider the summand

$$(25) \quad (kx) \otimes_k M_1 \otimes_k (kx) = xM_1x \cong M_1,$$

of B , and the nonunital subalgebra of B it generates, which we name

$$(26) \quad A = [k]\langle xM_1x \rangle.$$

Clearly, when we multiply (25) by itself an arbitrary positive number of times, there are no cases of a tensor product ending in M_1 being multiplied by one beginning with M_1 ; so the product takes the form

$$(27) \quad (kx)M_1(kx^2)M_1(kx^2)\cdots(kx^2)M_1(kx) \cong M_1 \otimes_k M_1 \otimes_k \cdots \otimes_k M_1 \quad (\text{with } \geq 1 \text{ } M_1 \text{ 's}).$$

(To see the isomorphism, note that $kx^2 \cong kx \cong k$ as k -modules, and $-\otimes_k k \otimes_k -$ simplifies to $-\otimes_k -$.) Thus (26) is, as claimed, isomorphic to the nonunital tensor algebra on the k -module M_1 .

Now suppose we multiply one of the summands (27) both on the left by a summand (23) and on the right by a summand (23). Again, because of the form of (27), this does not lead to an M_1 being multiplied by another M_1 , so the product always lies in a single summand (23). The reader should verify that this summand will again have the form (27) if and only if the left factor and the right factor are each either k or of the form (27). Thus, the summands (27) form something like an isolated subsemigroup among the summands (23); though we can't quite use that concept, since the summands (23) don't form a semigroup in a natural way, in view of (24).

We can now reason as in the proof of Lemma 9: given an ideal $I \subseteq A$, let J be the ideal of B that it generates. The general element of J can be written in the form (9), i.e., $\sum_{i=1}^n t_i f_i t'_i$, where each $f_i \in I$,

while each t_i and each t'_i lies in a summand (23). Those terms of (9) where both t_i and t'_i lie in summands that are either k or of the form (27), and so belong to $k + A$, will again belong to I , while all other summands will, by the result of the preceding paragraph, have values in the k -submodule of B spanned by the summands (23) not of the form (27). Hence if (9) lies in A , the sum of all terms of the latter sort must be zero. Hence our expression (9) will equal the sum of the terms of the first sort, hence lie in I , establishing the ideal extension property.

It is also easy to verify that for I and J as above, elements of J have zero components in the summands k , M_1 , and $kx + kx^2 + M_2$ of B . Thus, the images of A_1 and A_2 in $R = B/J$ are faithful.

Now from the assumption that A_0 can be generated as a *unital* k -algebra by a homomorphic image of M_1 , it is easy to see that the kernel of its augmentation map – let us call that kernel M_0 – can be generated as a *nonunital* k -algebra by such an image, hence, since A is isomorphic to the nonunital tensor algebra on M_1 , this algebra M_0 is isomorphic to A/I for some ideal $I \subseteq A$. By the preceding arguments, A/I embeds in the k -algebra $R = B/J$, where $J = BIB$. We thus have $A_0 = k + M_0 \cong k + A/I \subseteq B/J$, giving the desired embedding of A_0 in an algebra generated by embedded copies of A_1 and A_2 . \square

The statement of Theorem 15 is not particularly elegant. (If we had assumed k a field, we could have dropped condition (i), which is automatic in that case, making the statement a little nicer. If, instead, we had worked with nonunital rings, we could have dropped both that and the augmentation assumption on A_0 , and also shortened the proof.) Nor can Theorem 15 claim to be the strongest possible result of this sort. E.g., with slightly different assumptions on A_1 and A_2 , we could have weakened the assumption that A_0 was generated by one image of M_1 to allow it to be generated by a countable family of such images.

However, the proof of the theorem, as given, illustrates nicely several techniques that can be used in such situations.

I do not know a way of avoiding the need for something like the assumption in that theorem that A_2 admit a homomorphism onto $k[x]/(x^3)$, even if k is a field. But there are no evident examples showing that embeddability fails without such an assumption; so let us ask the following question. (Note that algebras are unital, since the contrary is not stated.)

Question 16. *Suppose k is a field, and A_1, A_2 are k -algebras, both of which have k -dimension ≥ 2 , and at least one of which has k -dimension ≥ 3 .*

Can every k -algebra A_0 with $\dim_k A_0 \leq \max(\aleph_0, \dim A_1, \dim A_2)$ be embedded in a k -algebra generated by an embedded copy of A_1 and an embedded copy of A_2 ?

The condition above that at least one of A_1, A_2 , have k -dimension ≥ 3 is needed, for if both are 2-dimensional, say with bases $\{1, b\}$ and $\{1, c\}$, then for each n , there are only $2n + 1$ alternating words of length $\leq n$ in b and c ; so $A_1 \amalg A_2$ has linear growth as a k -algebra. Hence no subalgebra of a homomorphic image of that coproduct can have faster than linear growth; so one cannot, for instance, embed the free algebra $k\langle x, y \rangle$ in such an algebra. With that case excluded, as in Question 16, $A_1 \amalg A_2$ is easily seen to have exponential growth, and, indeed, to contain free k -algebras on two generators, which in turn contain free k -algebras on countably many generators. If we could show that $A_1 \amalg A_2$ had a free subalgebra on two generators which satisfied the ideal extension property in $A_1 \amalg A_2$, then we would get a positive answer to Question 16 for countable-dimensional A_0 .

The case where k is not a field is messier; in particular, the module-theoretic condition (13) in Theorem 14 definitely cannot be dropped. For instance, if $k = \mathbb{Z}$, then the k -algebras \mathbb{Q} and $\mathbb{Z} + (\mathbb{Q}/\mathbb{Z})$, where the latter denotes the result of making \mathbb{Q}/\mathbb{Z} a nonunital \mathbb{Z} -algebra via the zero multiplication, and then adjoining a unit, cannot lie in a common unital \mathbb{Z} -algebra, since a \mathbb{Q} -algebra cannot have additive torsion – though $\mathbb{Z} + (\mathbb{Q}/\mathbb{Z})$ is generated as a unital \mathbb{Z} -algebra by a module-theoretic image of \mathbb{Q} .

6. CONSTRUCTIONS NOT USING GENERATORS AND RELATIONS

When one wants to establish that certain relations in an algebra do not entail other relations, an alternative to directly calculating the consequences of the relations is to construct an *action* of such an algebra exhibiting the non-equality (cf. discussion in §11.2 of [2]). Proofs of this sort are very convenient when they are available.

In fact, the first result of which I am aware showing that countably generated rings could be embedded *unitally* in finitely generated rings, the main theorem of [12] (which in fact gives embeddings in 2-generated rings), uses a technique of this sort, formulated in terms of infinite column-finite matrices over the given ring. The method is equally applicable to algebras. (A generalization to topological algebras is given in [7].)

Our final result, below shows how one of the results of the present note, the “ $xy^n z$ ” case of Theorem 4, can be given an alternative proof of this sort.

Note first that any k -algebra A can be embedded in the endomorphism algebra of some k -module (namely, any faithful A -module, regarded as a k -module), and that by taking a countably infinite direct sum of copies of such a module, we get another such k -module which, moreover, is a countably infinite direct sum of isomorphic copies of itself. We can now very quickly prove our result.

Lemma 17. *Let k be a commutative ring, and M a k -module which is a countably infinite direct sum of isomorphic copies of itself,*

$$(28) \quad M = \bigoplus_{i=0}^{\infty} M_i.$$

We shall write elements of $\text{End}_k(M)$ to the left of their arguments.

Then for every countable family $s_0, s_1, \dots, s_n, \dots$ of members of $\text{End}_k(M)$, there exist $x, y, z \in \text{End}_k(M)$ satisfying $xy^i z = s_i$, i.e., (1).

Hence for any countably generated k -algebra A , letting N be a faithful A -module, and applying the above to a direct sum M of a countably infinite family of copies of N , we recover the case of Theorem 4 that uses the relations (1).

Proof. Given (28), let $z \in \text{End}_k(M)$ carry M isomorphically to its submodule M_0 , and let $y \in \text{End}_k(M)$ take each M_i isomorphically to M_{i+1} . Viewing $y^i z$ as an isomorphism $M \rightarrow M_i$ for each i , let $x : M = \bigoplus_{i=0}^{\infty} M_i \rightarrow M$ be the map which acts on each M_i by $s_i(y^i z)^{-1}$. Then for each i we have $xy^i z = s_i$, as claimed. Letting R be the k -subalgebra of $\text{End}_k(M)$ generated by x, y and z , we get the desired case of Theorem 4. \square

REFERENCES

- [1] Jason P. Bell, Lance W. Small, and Agata Smoktunowicz, *Primitive algebraic algebras of polynomially bounded growth*, pp. 41–52 in *New Trends in Noncommutative Algebra*, Contemporary Mathematics, v.562, ed. P. Ara et al., 2012. <http://www.ams.org/books/conm/562/>. MR2905552.
- [2] George M. Bergman, *The diamond lemma for ring theory*, *Advances in Mathematics* **29** (1978) 178–218. MR **81b**:16001.
- [3] George M. Bergman and Nazih Nahlus, *Homomorphisms on infinite direct product algebras, especially Lie algebras*, *J. Algebra* **333** (2011) 67–104. MR **2012f**:17005.
- [4] L. A. Bokut', *Imbeddings into simple associative algebras* (Russian), *Algebra i Logika* **15** (1976) 117–142, 245. *Translated in Algebra and Logic* **15** (1976) 73–90, [http://link.springer.com/content/pdf/10.1007.BF01877233](http://link.springer.com/content/pdf/10.1007/BF01877233). MR **58#**:22167.
- [5] L. A. Bokut, Yuqun Chen and Qihui Mo, *Gröbner-Shirshov bases and embeddings of algebras*, *Internat. J. Algebra Comput.* **20** (2010) 875–900. MR **2011m**:16045. <http://arxiv.org/abs/0908.1992>.
(Note: the restrictions on $\text{card } k$ in some or all of the results of [5] will be removed by the results of [11]. I do not know whether [11] will also treat the corresponding restrictions in [4].)
- [6] L. A. Bokut' and P. S. Kolesnikov, *Gröbner-Shirshov bases: from inception to the present time* (Russian), *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **272** (2000) 26–67. *Translated in J. Math. Sci.* **116** (2003) 2894–2916. MR **2002a**:16030.
- [7] Antonio Fernández and Vladimír Müller, *Embedding of a separable locally pseudoconvex algebra into a locally pseudoconvex algebra with two generators*, *Topological, algebraical and combinatorial structures. Frolík's memorial volume. Discrete Math.* **108** (1992) 25–30. MR **94a**:46057.
- [8] George Grätzer *Universal algebra*, revised reprint of the 1979 second edition. Springer, New York, 2008. MR **2009i**:08001.
- [9] Serge Lang, *Algebra*, Addison-Wesley, third edition, 1993, reprinted as Springer G.T.M., v.211, 2002. MR **2003e**:00003.
- [10] A. I. Mal'cev, *On a representation of nonassociative rings* (Russian), *Uspehi Matem. Nauk (N.S.)* **7** (1952) 181–185. <http://mi.mathnet.ru/umn8290>. MR **13**,816e.
- [11] Qihui Mo, paper in preparation.
- [12] K. C. O'Meara, C. I. Vinsonhaler and W. J. Wickless, *Identity-preserving embeddings of countable rings into 2-generator rings*, *Rocky Mountain J. Math.* **19** (1989) 1095–1105. <http://dx.doi.org/10.1216/RMJ-1989-19-4-1095>. MR **91h**:16057.
- [13] A. I. Shirshov, *Some theorems on immersion for rings* (Russian), *Mat. Sb. N.S.* **40** (82) (1956) 65–72. MR **19**,525c.

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