AFFINE KAC-MOODY ALGEBRAS, INTEGRABLE SYSTEMS AND THEIR DEFORMATIONS

EDWARD FRENKEL

Representation theory of affine Kac-Moody algebras at the critical level contains many intricate structures, in particular, the hamiltonian structures of the KdV and modified KdV hierarchies and the Miura transformation between them. In this talk I will describe these structures and their deformations which will lead us to the deformed Virasoro and $\mathcal{W}$–algebras and the integrable hierarchies associated to them. I will also discuss briefly the relation of these matters to the geometric Langlands correspondence.

It is a great honor for me to give this talk as the first recipient of the Hermann Weyl Prize. Weyl was a pioneer of applications of symmetry in quantum physics, a scientist who truly appreciated the beauty of mathematics. He once said: *My work has always tried to unite the true with the beautiful and when I had to choose one or the other, I usually chose the beautiful.*

1. Affine Kac-Moody algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Fix an invariant inner product κ on $\mathfrak{g}$ and let $\widehat{\mathfrak{g}}_\kappa$ denote the one-dimensional central extension of $\mathfrak{g} \otimes \mathbb{C}((t))$ with the commutation relations

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\kappa(A, B) \operatorname{Res} f dg)K, \tag{1}$$

where K is the central element. The Lie algebra $\widehat{\mathfrak{g}}_\kappa$ is the *affine Kac-Moody algebra* associated to $\mathfrak{g}$.

A representation of $\widehat{\mathfrak{g}}_\kappa$ on a complex vector space V is called *smooth* if for any vector $v \in V$ there exists $N \in \mathbb{Z}_+$ such that $\mathfrak{g} \otimes t^N \mathbb{C}[[t]] \cdot v = 0$. We also require that K acts on V as the identity (since the space of invariant inner products on $\mathfrak{g}$ is one-dimensional, this is equivalent to a more traditional approach whereby one fixes κ but allows K to act as the identity times a scalar, called the level).

Let $U_\kappa(\widehat{\mathfrak{g}})$ be the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ of $\widehat{\mathfrak{g}}_\kappa$ by the ideal generated by $(K - 1)$. Define its completion $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ as follows:

$$\widetilde{U}_\kappa(\widehat{\mathfrak{g}}) = \varprojlim U_\kappa(\widehat{\mathfrak{g}})/U_\kappa(\widehat{\mathfrak{g}}) \cdot (\mathfrak{g} \otimes t^N \mathbb{C}[[t]]).$$

It is clear that $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ is a topological algebra which acts on all smooth representations of $\widehat{\mathfrak{g}}_\kappa$, on which K acts as the identity. We shall recall the description of the center $Z(\widehat{\mathfrak{g}})$ of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ from [FF3, F3].

First we need to introduce the notion of an oper.

The text of the Hermann Weyl Prize lecture given at the XXIV Colloquium on Group Theoretical Methods in Physics, Paris, July 17, 2002. To appear in the Proceedings of the Colloquium.

2. Opers

We start with the definition of an $\mathfrak{sl}_n$–oper. Let X be a smooth algebraic curve and Ω the line bundle of holomorphic differentials on X. We will fix a square root $\Omega^{1/2}$ of Ω. An $\mathfrak{sl}_n$–oper on X is an nth order differential operator acting from the holomorphic sections of $\Omega^{-(n-1)/2}$ to those of $\Omega^{(n+1)/2}$ whose principal symbol is equal to 1 and subprincipal symbol is equal to 0 (note that for these conditions to be coordinate-independent, this operator must act from $\Omega^{-(n-1)/2}$ to $\Omega^{(n+1)/2}$). Locally, we can choose a coordinate t and write this operator as

$$L = \partial_t^n + v_1(t)\partial_t^{n-2} + \ldots + v_{n-1}(t). \tag{2}$$

It is not difficult to obtain the transformation formulas for the coefficients $v_1(t), \ldots, v_{n-1}(t)$ of this operator under changes of coordinates.

For example, an $\mathfrak{sl}_2$–oper is nothing but a Sturm-Liouville operator of the form $\partial_t^2 + v(t)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$. Under the change of coordinates $t = \varphi(s)$ we have the following transformation formula

$$v \mapsto \widetilde{v}, \qquad \widetilde{v}(s) = v(\varphi(s))\left(\varphi'(s)\right)^2 + \frac{1}{2}\{\varphi, s\},$$

where

$$\{\varphi, s\} = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2$$

is the Schwarzian derivative. Operators of this form are known as *projective connections* (see, e.g., [FB], Sect. 7.2).

Drinfeld and V. Sokolov [DS] have introduced an analogue of operators (2) for a general simple Lie algebra $\mathfrak{g}$. Their idea was to replace the operator (2) by the first order matrix differential operator

$$\partial_t + \begin{pmatrix} 0 & v_1 & v_2 & \cdots & v_{n-1} \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \tag{3}$$

Now consider the space of more general operators of the form

$$\partial_t + \begin{pmatrix} * & * & * & \cdots & * \\ -1 & * & * & \cdots & * \\ 0 & -1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & * \end{pmatrix} \tag{4}$$

The group of upper triangular matrices with 1's on the diagonal acts on this space by gauge transformations

$$\partial_t + A(t) \mapsto \partial_t + gA(t)g^{-1} - g^{-1}\partial_t g.$$

It is not difficult to show that this action is free and each orbit contains a unique operator of the form (3). Therefore, locally, over a sufficiently small neighborhood U

of X, equipped with a coordinate t, the space of $\mathfrak{sl}_n$–opers on U may be identified with the quotient of the space of operators of the form (4) by the gauge action of the group of upper triangular matrices.

Now let us generalize the latter definition to the case of an arbitrary simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be the Cartan decomposition of $\mathfrak{g}$ and e_i, h_i and $f_i, i = 1, \ldots, \ell$, be the Chevalley generators of $\mathfrak{n}_+, \mathfrak{h}$ and $\mathfrak{n}_-$, respectively. Then the analogue of the space of operators of the form (4) is the space of operators

$$\partial_t - \sum_{i=1}^{\ell} f_i + \mathbf{v}(t), \qquad \mathbf{v}(t) \in \mathfrak{b}_+, \tag{5}$$

where $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$. This space is preserved by the action of the group of N_+–valued gauge transformations, where N_+ is the Lie group corresponding to $\mathfrak{n}_+$. Following [DS], we define the space of $\mathfrak{g}$–opers over a sufficiently small neighborhood U of X, equipped with a coordinate t, as the quotient of the space of all operators of the form (5) by the N_+–valued gauge transformations. One shows that these gauge transformations act freely, and one can find canonical representatives of each orbit labeled by ℓ–tuples of functions $v_1(t), \ldots, v_\ell(t)$. The first of them, $v_1(t)$, transforms as a projective connection, and $v_i(t), i > 1$, transforms as a k_i–differential (i.e., a section of Ω^{k_i}), where $\{k_i\}_{i=1,\ldots,\ell}$ are the orders of the Casimir elements of $\mathfrak{g}$, i.e., the generators of the center of $U(\mathfrak{g})$. Using these transformations, one glues together the spaces of $\mathfrak{g}$–opers on different open subsets of X and thus obtains the notion of a $\mathfrak{g}$–oper on X. A. Beilinson and V. Drinfeld have given a more conceptual definition of opers on X as G–bundles with a connection and a reduction to the Borel subgroup B_+ (the Lie group of $\mathfrak{b}_+$) satisfying a certain transversality condition, see [BD], Sect. 3.

Denote by $\mathrm{Op}_{\mathfrak{g}}(D^\times)$ the space of $\mathfrak{g}$–opers on the (formal) punctured disc. This is the quotient of the space of operators of the form (5), where $\mathbf{v}(t) \in \mathfrak{b}_+((t))$, by the gauge action of $N_+((t))$.

Drinfeld and Sokolov have obtained $\mathrm{Op}_{\mathfrak{g}}(D^\times)$ as the result of the hamiltonian reduction of the space of all operators of the form $\partial_t + A(t), A(t) \in \mathfrak{g}((t))$. The latter space may be identified with a hyperplane in the dual space to the affine Lie algebra $\widehat{\mathfrak{g}}_{\nu_0}, \nu_0 \neq 0$, which consists of all linear functionals taking value 1 on K. It carries the Kirillov-Kostant Poisson structure; the non-zero invariant inner product ν_0 on $\mathfrak{g}$ appears as a parameter of this Poisson structure.

Applying the Drinfeld-Sokolov reduction, we obtain a Poisson structure on the algebra $\mathrm{Fun}(\mathrm{Op}_{\mathfrak{g}}(D^\times))$ of functions on $\mathrm{Op}_{\mathfrak{g}}(D^\times)$. This Poisson algebra is called the *classical* $\mathcal{W}$*–algebra* associated to $\mathfrak{g}$. In the case when $\mathfrak{g} = \mathfrak{sl}_n$, this Poisson structure is the (second) *Adler–Gelfand–Dickey* Poisson structure. Actually, it is a member of a two-dimensional family of Poisson structures on $\mathrm{Op}_{\mathfrak{sl}_n}(D^\times)$ with respect to which the flows of the nth KdV hierarchy are hamiltonian. We recall that in terms of the first description of $\mathfrak{sl}_n$–opers, i.e., as operators L of the form (2), the KdV equations may be written in the Lax form

$$\partial_{t_m} L = [(L^{m/n})_+, L], \qquad m > 0, m \not| \, n, \tag{6}$$

where $L^{1/n} = \partial_t + \ldots$ is the pseudodifferential operator obtained by extracting the nth root of L and $+$ indicates the differential part of a pseudodifferential operator.

Drinfeld and Sokolov have defined an analogue of the KdV hierarchy on the space of $\mathfrak{g}$–opers for an arbitrary $\mathfrak{g}$. The equations of this hierarchy are hamiltonian with respect to the above Poisson structure (in fact, they are bihamiltonian, but we will not discuss here the other hamiltonian structure).

3. The center

Let us go back to the completed universal enveloping algebra $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ and its center $Z_\kappa(\widehat{\mathfrak{g}})$. Note that $Z_\kappa(\widehat{\mathfrak{g}})$ is a Poisson algebra. Indeed, choosing a non-zero invariant inner product κ_0, we may write a one-parameter deformation of κ as $\kappa + \epsilon\kappa_0$. Given two elements, $A, B \in Z_\kappa(\widehat{\mathfrak{g}})$, we consider their arbitrary ϵ–deformations, $A(\epsilon), B(\epsilon) \in \widetilde{U}_{\kappa+\epsilon\kappa_0}(\widehat{\mathfrak{g}})$. Then the ϵ–expansion of the commutator $[A(\epsilon), B(\epsilon)]$ will not have a constant term, and its ϵ–linear term, specialized at $\epsilon = 0$, will again be in $Z_\kappa(\widehat{\mathfrak{g}})$ and will be independent of the deformations of A and B. Thus, we obtain a bilinear operation on $Z_\kappa(\widehat{\mathfrak{g}})$, and one checks that it satisfies all properties of a Poisson bracket.

Now we can describe the Poisson algebra $Z_\kappa(\widehat{\mathfrak{g}})$. For a simple Lie algebra $\mathfrak{g}$ we denote by ${}^L\mathfrak{g}$ its Langlands dual Lie algebra, whose Cartan matrix is the transpose of that of $\mathfrak{g}$ (note that this duality only affects the Lie algebras of series B and C, which get interchanged). We have a canonical identification ${}^L\mathfrak{h} = \mathfrak{h}^*$.

Let κ_c be the critical inner product on $\mathfrak{g}$ defined by the formula

$$\kappa_c(x, y) = -\frac{1}{2} \operatorname{Tr}_{\mathfrak{g}} \operatorname{ad} x \operatorname{ad} y.$$

In the standard normalization of [K], the modules over $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ on which K acts as the identity are the $\widehat{\mathfrak{g}}$–modules of *critical level* $-h^\vee$, where $h^\vee$ is the dual Coxeter number.

Theorem 1 ([FF3, F3])**.**
(1) *If* $\kappa \neq \kappa_c$*, then* $Z_\kappa(\widehat{\mathfrak{g}}) = \mathbb{C}$.
(2) The center $Z_{\kappa_c}(\widehat{\mathfrak{g}})$ *is isomorphic, as a Poisson algebra, to the classical* $\mathcal{W}$*–algebra* $\operatorname{Fun}(\operatorname{O}p_{{}^L\mathfrak{g}}(D^\times))$.

Thus, we recover the (second) Poisson structure of the ${}^L\mathfrak{g}$–KdV hierarchy from the center of the completed universal enveloping algebra $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$. Note that the two Poisson structures appearing in the theorem depend on parameters: the inner products κ_0 on $\mathfrak{g}$ and ν_0 on ${}^L\mathfrak{g}$. In the above isomorphism they have to agree in the obvious sense, namely, that the restriction of κ_0 to $\mathfrak{h}$ is dual to the restriction of ν_0 to ${}^L\mathfrak{h} = \mathfrak{h}^*$.

For example, $\operatorname{Op}_{\mathfrak{sl}_2}(D^\times) = \{\partial_t^2 - v(t)\}$, where $v(t) = \sum_{n\in\mathbb{Z}} v_n t^{-n-2}$ is a formal Laurent series. Therefore $\operatorname{Fun}(\operatorname{Op}_{\mathfrak{sl}_2}(D^\times))$ is a completion of the polynomial algebra $\mathbb{C}[v_n]_{n\in\mathbb{Z}}$. The Poisson structure is that of the classical Virasoro algebra; it is uniquely determined by the Poisson brackets between the generators

$$\{v_n, v_m\} = (n - m)v_{n+m} - \frac{1}{2}(n^3 - n)\delta_{n,-m}.$$

Under the above isomorphism, the generators v_n are mapped to the *Segal-Sugawara operators* S_n. Those are defined (for an arbitrary $\mathfrak{g}$) by the formula

$$S(z) = \sum_{n\in\mathbb{Z}} S_n\, z^{-n-2} = \sum_a {:}J^a(z)^2{:},$$

where

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n \, z^{-n-1}, \qquad J^a_n = J^a \otimes t^n,$$

and $\{J^a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to κ_0.

For $\mathfrak{g} = \mathfrak{sl}_2$, the center $Z_{\kappa_c}(\widehat{\mathfrak{sl}}_2)$ is a completion of the polynomial algebra generated by $S_n, n \in \mathbb{Z}$. For a general $\mathfrak{g}$, we also have $\ell - 1$ "higher" Segal–Sugawara operators $S^{(i)}_n, i = 2, \ldots, \ell, n \in \mathbb{Z}$, of orders equal to the orders of the Casimirs of $\mathfrak{g}$, and the center $Z_{\kappa_c}(\widehat{\mathfrak{g}})$ is a completion of the algebra of polynomials in these operators. However, explicit formulas for $S^{(i)}_n$ with $i > 1$ are unknown in general.

4. Miura transformation

In addition to operators of the form (3), it is useful to consider the operators

$$\partial_t + \begin{pmatrix} u_1 & 0 & 0 & \cdots & 0 \\ -1 & u_2 & 0 & \cdots & 0 \\ 0 & -1 & u_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & u_n \end{pmatrix}, \qquad \sum_{i=1}^n u_i = 0. \tag{7}$$

It is easy to see that the operator (7) defines the same oper as the operator (3) (i.e., that they are gauge equivalent under the action of the group of upper triangular matrices) if and only if we have the following identity:

$$\partial_t^n + v_1(t)\partial_t^{n-2} + \ldots + v_{n-1}(t) = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t)), \tag{8}$$

This equation expresses $v_1, \ldots, v_{n-1}$ as differential polynomials in $u_1, \ldots, u_\ell$. For example, for $n = 2$ we have

$$\partial_t^2 - v = (\partial_t - u)(\partial_t + u), \qquad \text{i.e.,} \qquad v = u^2 - u'. \tag{9}$$

The latter formula is known as the *Miura transformation*. R. Miura had found that this formula relates solutions of the KdV equation to solutions of another soliton equation, called the modified KdV, or mKdV, equation. This suggested that the KdV equation has something to do with the second order operators $\partial_t^2 - v$, because this formula appears in the splitting of this operator into two operators of order one. It was this observation that has subsequently led to the discovery of the inverse scattering method (in a subsequent work by Gardner, Green, Kruskal and Miura).

The Miura transformation may be viewed as a map from the space of the first order operators $\{\partial_t + u(t)\}$ to the space $\mathrm{Op}_{\mathfrak{sl}_2}(D^\times) = \{\partial_t^2 - v(t)\}$ of $\mathfrak{sl}_2$–opers (or projective connections) on $D^\times$. In order to make it coordinate-independent, we must view the operator $\partial_t + u(t)$ as acting from $\Omega^{-1/2}$ to $\Omega^{1/2}$, i.e., consider it as a connection on the line bundle $\Omega^{-1/2}$. Denote the space of such connections on $D^\times$ by $\mathrm{Conn}_{\mathfrak{sl}_2}(D^\times)$. Then this map is actually a Poisson map $\mathrm{Conn}_{\mathfrak{sl}_2}(D^\times) \to \mathrm{Op}_{\mathfrak{sl}_2}(D^\times)$ if we introduce the Poisson structure $\mathrm{Conn}_{\mathfrak{sl}_2}(D^\times)$ by the formula

$$\{v_n, v_m\} = \frac{1}{2} n \delta_{n,-m}, \tag{10}$$

where $u(t) = \sum_{n\in\mathbb{Z}} u_n t^{-n-1}$. Thus, the algebra of functions on $\mathrm{Conn}_{\mathfrak{sl}_2}(D^\times)$, which is a completion of the polynomial algebra $\mathbb{C}[u_n]_{n\in\mathbb{Z}}$, is a Heisenberg–Poisson algebra.

Drinfeld and Sokolov have defined an analogue of the Miura transformation for an arbitrary simple Lie algebra $\mathfrak{g}$. The role of the operator $\partial_t + u(t)$ is now played by the operator $\partial_t + \mathbf{u}(t)$, where $\mathbf{u}(t)$ takes values in $\mathfrak{h}((t))$, considered as a connection on the H–bundle $\Omega^{\rho^\vee}$. In other words, under the change of variables $t = \varphi(s)$ it transforms as follows:

$$u \mapsto \widetilde{u}, \qquad \widetilde{u}(s) = u(\varphi(s))\varphi'(s) - \rho^\vee \left(\frac{\varphi''(s)}{\varphi'(s)}\right).$$

Denote the space of such operators on the punctured disc by $\mathrm{Conn}_{\mathfrak{g}}(D^\times)$. Then we have a natural map

$$\mu : \mathrm{Conn}_{\mathfrak{g}}(D^\times) \to \mathrm{Op}_{\mathfrak{g}}(D^\times)$$

which sends $\partial_t + \mathbf{u}(t)$ to the oper which is the gauge class of the operator $\partial_t - \sum_{i=1}^{\ell} f_i + \mathbf{u}(t)$. This is the Miura transformation corresponding to the Lie algebra $\mathfrak{g}$.

Define a Poisson structure on $\mathrm{Conn}_{\mathfrak{g}}(D^\times)$ as follows. Write $u_i(t) = \langle \alpha_i, \mathbf{u}(t)\rangle$ and $u_i(t) = \sum_{n\in\mathbb{Z}} u_{i,n} t^{-n-1}$. Then set

$$\{u_{i,n}, u_{j,m}\} = n\nu_0^{-1}(\alpha_i, \alpha_j)\delta_{n,-m},$$

where ν_0^{-1} is the inner product on $\mathfrak{h}^*$ induced by $\nu_0|_{\mathfrak{h}}$. Then the Miura transformation $\mathrm{Conn}_{\mathfrak{g}}(D^\times) \to \mathrm{Op}_{\mathfrak{g}}(D^\times)$ is a Poisson map (if we take the Poisson structure on $\mathrm{Op}_{\mathfrak{g}}(D^\times)$ corresponding to ν_0).

Drinfeld and Sokolov [DS] have defined the modified KdV hierarchy corresponding to $\mathfrak{g}$ on $\mathrm{Conn}_{\mathfrak{g}}(D^\times)$. The equations of this $\mathfrak{g}$–mKdV hierarchy are hamiltonian with respect to the above Poisson structure. The Miura transformation intertwines the $\mathfrak{g}$–mKdV and $\mathfrak{g}$–KdV hierarchies.

5. Wakimoto modules

In Theorem 1 we identified the Poisson structure of the ${}^L\mathfrak{g}$–KdV hierarchy with the Poisson structure on the center $Z_{\kappa_c}(\widehat{\mathfrak{g}})$. It is natural to ask whether one can interpret in a similar way the Poisson structure of the ${}^L\mathfrak{g}$–mKdV hierarchy and the Miura transformation. This can indeed be done using the Wakimoto modules of critical level constructed in [W, FF1, FF2, F3].

Let us briefly explain the idea of the construction of the Wakimoto modules (see [FB], Ch. 10-11, and [F3] for more details). Set $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$. Given a linear functional $\chi : \mathfrak{h}((t)) \to \mathbb{C}$, we extend it trivially to $\mathfrak{n}_-((t))$ and obtain a linear functional on $\mathfrak{b}_-((t))$, also denoted by χ. Let $\mathbb{C}_\chi$ be the corresponding one-dimensional representation of $\mathfrak{b}_-((t))$. We would like to associate to it a smooth representation of $\widehat{\mathfrak{g}}$. It is clear that the induced module $\mathrm{Ind}_{\mathfrak{b}_-((t))}^{\mathfrak{g}((t))} \mathbb{C}_\chi$ is not smooth. Therefore we need to modify the construction of induction corresponding to a different choice of vacuum. In the induced module the vacuum is annihilated by the Lie subalgebra $\mathfrak{n}_-((t))$, while in the Wakimoto module obtained by the "semi-infinite" induction the vacuum is annihilated by $t\mathfrak{g}[t] \oplus \mathfrak{n}_+$.

However, when one applies the "semi-infinite" induction procedure one has to deal with certain "quantum corrections". The effect of these corrections is two-fold: first of all, the resulting module is a module over the central extension of $\mathfrak{g}((t))$, i.e., the affine algebra $\widehat{\mathfrak{g}}$, of critical level. Second, the parameters of the module no longer behave as linear functionals on $\mathfrak{h}((t))$, or, equivalently, as elements of the space $\mathfrak{h}^*((t))dt = {}^L\mathfrak{h}((t))dt$ of $\mathfrak{h}^*$–valued one-forms on $D^\times$, but as connections on the ${}^LH$–bundle Ω^ρ. They are precisely the elements of the space $\mathrm{Conn}_{^L\mathfrak{g}}(D^\times)$ which is a principal homogeneous space over ${}^L\mathfrak{h}((t))dt$.

Thus we obtain a family of smooth representations of $\widetilde{U}_{\kappa_c}(\widehat{\mathfrak{g}})$ (on which the central element K acts as the identity) parameterized by points of $\mathrm{Conn}_{^L\mathfrak{g}}(D^\times)$. These are the *Wakimoto modules* of critical level. For $\chi \in \mathrm{Conn}_{^L\mathfrak{g}}(D^\times)$ we denote the corresponding module by W_χ.

Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ with the standard basis $\{e, h, f\}$. Consider the Weyl algebra with generators $a_n, a^*_n, n \in \mathbb{Z}$, and relations $[a_n, a^*_m] = \delta_{n,-m}$. Let M be the Fock representation generated by a vector $|0\rangle$ such that $a_n|0\rangle = 0, n \geq 0$ and $a^*_n|0\rangle = 0, n > 0$. Set $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, etc. Then for any Laurent series $u(t)$ the formulas

$$\begin{aligned}
e(z) &= a(z),\\
h(z) &= -2{:}a(z)a^*(z){:} + u(z),\\
f(z) &= -{:}a(z)a^*(z)^2{:} + u(z)a^*(z) - 2\partial_z a^*(z)
\end{aligned}$$

define an $\widehat{\mathfrak{sl}}_2$–module structure on M. This is the Wakimoto module attached to $\partial_t - u(t)$. One checks easily that in order for the h_n's to transform as the functions t^n (or, equivalently, for $h(z)dz$ to transform as a one-form), $\partial_t - u(t)$ needs to transform as a connection on $\Omega^{-1/2}$.

One also checks that the Segal-Sugawara operator $S(z)$ acts on this module as $u^2 - u'$, i.e., through the Miura transformation. This statement has the following generalization for an arbitrary $\mathfrak{g}$.

Theorem 2 ([FF3, F3])**.** *The center $Z_{\kappa_c}(\widehat{\mathfrak{g}})$ acts on $W_\chi, \chi \in \mathrm{Conn}_{^L\mathfrak{g}}(D^\times)$, according to a character. The corresponding point in $\mathrm{Spec}\, Z_{\kappa_c}(\widehat{\mathfrak{g}}) = \mathrm{Op}_{^L\mathfrak{g}}(D^\times)$ is $\mu(\chi)$, where $\mu : \mathrm{Conn}_{^L\mathfrak{g}}(D^\times) \to \mathrm{Op}_{^L\mathfrak{g}}(D^\times)$ is the Miura transformation.*

Thus, we obtain an interpretation of the Miura transformation as an affine analogue of the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^W$. We remark that the map $\mathrm{Conn}_{^L\mathfrak{g}}(D^\times) \to \mathrm{Spec}\, Z_{\kappa_c}(\widehat{\mathfrak{g}})$ that we obtain this way is manifestly Poisson because the Wakimoto modules may be deformed away from the critical level (this deformation gives rise to a Poisson structure on $\mathrm{Conn}_{^L\mathfrak{g}}(D^\times)$ which coincides with the one introduced above).

In summary, we have now described the phase spaces of both the generalized KdV and mKdV hierarchies, their Poisson structures and a map between them in terms of representations of affine Kac-Moody algebras of critical level.

6. Local Langlands correspondence for affine algebras

The classical local Langlands correspondence aims to describe the isomorphism classes of smooth representations of $G(F)$, where G is a reductive algebraic group and $F = \mathbb{Q}_p$ or $\mathbb{F}_q((t))$, in terms of homomorphisms from the Galois group of F to the Langlands dual group ${}^LG$ (this is the group for which the sets of characters and cocharacters of the maximal torus are those of G, interchanged).

Let us replace $\mathbb{F}_q((t))$ by $\mathbb{C}((t))$ and G by its Lie algebra $\mathfrak{g}$. Then we try to describe smooth representations of the central extension of the loop algebra $\mathfrak{g}((t))$ in terms of some Galois data. But in the geometric context the Galois group should be thought of as a sort of fundamental group. Hence we replace the notion of a Galois representation by the notion of a ${}^LG$–local system, or equivalently, a ${}^LG$–bundle with connection on $D^\times$.

The local Langlands correspondence in this context should be a statement that to each ${}^LG$–bundle with connection on $D^\times$ corresponds a category of $\widehat{\mathfrak{g}}$–modules. Here is an example of such a statement in which the Wakimoto modules of critical level introduced in the previous section play an important role (this is part of an ongoing joint project with D. Gaitsgory).

First we introduce the notion of *nilpotent opers*. Those are roughly those opers on $D^\times$ which have regular singularity at the origin and unipotent monodromy around 0. We denote the space of nilpotent $\mathfrak{g}$–opers by $\mathrm{nOp}_{\mathfrak{g}}$. For $\mathfrak{g} = \mathfrak{sl}_2$, its points are the projective connections of the form $\partial_z^2 - v(z)$, where $v(t) = \sum_{n \leq -1} v_n t^{-n-2}$. We have a residue map $\mathrm{Res} : \mathrm{nOp}_{\mathfrak{g}} \to \mathfrak{n}$ which for $\mathfrak{g} = \mathfrak{sl}_2$ takes the form $\partial_t^2 - v(t) \mapsto v_{-1}$.

Recall that for a nilpotent element $x \in {}^L\mathfrak{g}$, the *Springer fiber* of x is the variety of all Borel subalgebras of ${}^L\mathfrak{g}$ containing x. For example, the Springer fiber at 0 is just the flag variety of ${}^L\mathfrak{g}$.

Lemma 1. *The set of points of the fiber $\mu^{-1}(\rho)$ of the Miura transformation over a nilpotent ${}^L\mathfrak{g}$–oper ρ is in bijection with the set of points of the Springer fiber of* $\mathrm{Res}(\rho) \in {}^L\mathfrak{n}$.

For example, the set of points of the fiber $\mu^{-1}(\rho)$ of the Miura transformation over a regular ${}^L\mathfrak{g}$–oper is the set of points of the flag variety of ${}^L\mathfrak{g}$.

Now fix $\rho \in \mathrm{nOp}_{{}^L\mathfrak{g}}$ and consider the category $\mathcal{C}_\rho$ of $\widehat{\mathfrak{g}}$–modules of critical level on which $Z_{\kappa_c}(\widehat{\mathfrak{g}}) \simeq \mathrm{Fun}(\mathrm{Op}_{{}^L\mathfrak{g}}(D^\times))$ acts through the central character $Z_{\kappa_c}(\widehat{\mathfrak{g}}) \to \mathbb{C}$ corresponding to ρ and such that the Lie subalgebra $(t\mathfrak{g}[[t]] \oplus \mathfrak{n}_+) \subset \widehat{\mathfrak{g}}$ acts locally nilpotently and the Lie subalgebra $\mathfrak{h}$ acts with integral generalized eigenvalues. Then, according to a conjecture of Gaitsgory and myself, the derived category of $\mathcal{C}_\rho$ is equivalent to the derived category of quasicoherent sheaves on the Springer fiber of $\mathrm{Res}(\rho)$ (more precisely, the corresponding DG-scheme).

In particular, under this equivalence the skyscraper sheaf at a point of the Springer fiber of $\mathrm{Res}(\rho)$, which is the same as a point χ of $\mathrm{Conn}_{{}^L\mathfrak{g}}(D^\times)$ projecting onto ρ under the Miura transformation, should correspond to the Wakimoto module W_χ. Thus, the above conjecture means that, loosely speaking, any object of the category $\mathcal{C}_\rho$ may be "decomposed" into a "direct integral" of Wakimoto modules.

7. A q–deformation

Now we wish to define q–deformations of the structures described in the previous sections. In particular, we wish to introduce q–analogues of opers and connections (together with their Poisson structures) and of the Miura transformation between them. We also wish to define q–analogues of the KdV hierarchies. For that we replace the universal enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{g}}$ by the corresponding *quantized enveloping algebra* $U_q(\widehat{\mathfrak{g}})$.

Then the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level (with its Poisson structure defined in the same way as in the undeformed case) should be viewed as q–analogue of the algebra of functions on opers (i.e., the classical $\mathcal{W}$–algebra). One the other hand, parameters of Wakimoto modules should be viewed as q–analogues of connections, and the action of the center of $U_q(\widehat{\mathfrak{g}})$ on Wakimoto modules should give us a q–analogue of the Miura transformation.

In [FR1], N. Reshetikhin and I have computed these structures in the case when $\mathfrak{g} = \mathfrak{sl}_n$ (we used the Wakimoto modules over $U_q(\widehat{\mathfrak{sl}}_n)$ constructed in [AOS]). Let us describe the results. The q–analogues of $\mathfrak{sl}_n$–opers are q–difference operators of the form

$$L_q = D^n + t_1(z)D^{n-1} + \ldots + t_{n-1}(z)D + 1, \tag{11}$$

where $(Df)(z) = f(zq^2)$. The q–analogues of connections are operators $D+\mathbf{\Lambda}(z)$, where $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ and $\prod_{i=1}^n \Lambda_i(z) = 1$. The q–analogue of the Miura transformation is the formula expressing the splitting of the operator (11) into a product of first order operators

$$D^n + t_1(z)D^{n-1} + \ldots + t_{n-1}(z)D + 1 = (D + \Lambda_1(z)) \ldots (D + \Lambda_n(z)).$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$ the q–Miura transformation is

$$t(z) = \Lambda(z) + \Lambda(zq^2)^{-1}. \tag{12}$$

Note that in the limit $q = e^h, h \to 0$ we have $t(z) = 2 + 4h^2 v(z) z^2 + \ldots$ and $\Lambda(z) = e^{2hu(z)z}$ so that we obtain the ordinary Miura transformation $v = u^2 - u'$.

The Poisson structures with respect to which this map is Poisson are given by the formulas [FR1]

$$\{\Lambda(z), \Lambda(w)\} = (q - q^{-1}) f(w/z)\, \Lambda(z)\Lambda(w), \qquad f(x) = \sum_{n \in \mathbb{Z}} \frac{q^n - q^{-n}}{q^n + q^{-n}}\, x^n,$$

$$\{t(z), t(w)\} = (q - q^{-1}) \left(f(w/z)\, t(z)t(w) + \delta\left(\frac{w}{zq^2}\right) - \delta\left(\frac{wq^2}{z}\right)\right),$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. Analogous formulas for $\mathfrak{sl}_n, n > 2$, may be found in [FR1].

The equations of the q–analogue of the nth KdV hierarchy are given by the formulas

$$\partial_{t_m} L_q = \left[(L_q^{m/n})_+, L_q\right], \qquad m > 0, m \not| \, n.$$

These equations are hamiltonian with respect to the above Poisson structure [F2].

For simple Lie algebras other than $\mathfrak{sl}_n$, Wakimoto modules have not yet been constructed. Nevertheless, in [FR2] we have generalized the above formulas to the case of an arbitrary $\mathfrak{g}$.

Another approach is to define q–analogues of the classical $\mathcal{W}$–algebras by means of a q–analogue of the Drinfeld-Sokolov reduction. This has been done in [FRS, SS].

If we write $\Lambda(z) = Q(zq^{-2})/Q(z)$, then formula (12) becomes

$$t(z) = \frac{Q(zq^2)}{Q(z)} + \frac{Q(zq^{-2})}{Q(z)},$$

which is the Baxter formula for the Bethe Ansatz eigenvalues of the transfer-matrix in the XXZ spin chain model. This is not coincidental. The transfer-matrices of spin models may be obtained from central elements of $U_q(\widehat{\mathfrak{g}})$ of critical level and the Bethe eigenvectors can be constructed using Wakimoto modules. Then the formula for the eigenvalues becomes precisely the formula for the q–Miura transformation (see [FR2]). In the quasi-classical limit we obtain that the formula for the Bethe Ansatz eigenvalues of the Gaudin model (the quasi-classical limit of the XXZ model) may be expressed via the (ordinary) Miura transformation formula. This was explained (in the case of an arbitrary simple Lie algebra $\mathfrak{g}$) in [FFR, F1].

8. The q–characters

Let Rep $U_q(\widehat{\mathfrak{g}})$ be the Grothendieck ring of finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. N.Reshetikhin and M.Semenov-Tian-Shansky [RS] have given an explicit construction of central elements of $U_q(\widehat{\mathfrak{g}})$ of critical level. It amounts to a homomorphism from Rep $U_q(\widehat{\mathfrak{g}})$ to $Z_q((z))$, where Z_q is the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level. Combining this homomorphism with the q–Miura transformation defined in [FR2], we obtain an injective homomorphism

$$\chi_q : \text{Rep } U_q(\widehat{\mathfrak{g}}) \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i=1,\ldots,\ell; a \in \mathbb{C}^\times}$$

which we call the q–character homomorphism (see [FR3]). It should be viewed as a q–analogue of the ordinary character homomorphism

$$\chi : \text{Rep } U(\mathfrak{g}) \to \mathbb{Z}[y_i^{\pm 1}]_{i=1,\ldots,\ell},$$

where the y_i's are the fundamental coweights of $\mathfrak{g}$. Under the forgetful homomorphism $Y_{i,a} \mapsto y_i$, the q–character of a $U_q(\widehat{\mathfrak{g}})$–module V becomes the ordinary character of the restriction of V to $U_q(\mathfrak{g})$ (specialized at $q = 1$). For instance, if $V(a)$ is the two-dimensional representation of $U_q(\widehat{\mathfrak{sl}}_2)$ with the evaluation parameter a, then $\chi_q(V(a)) = Y_a + Y_{aq^2}^{-1}$, which under the forgetful homomorphism become the character $y + y^{-1}$ of the two-dimensional representation of $U_q(\mathfrak{sl}_2)$ (or $U(\mathfrak{sl}_2)$). In [FM2], the notion of q–characters was extended to the case when q is a root of unity.

When $\mathfrak{g} = \mathfrak{sl}_n$, the variables $Y_{i,a}$ correspond to the series $\Lambda_i(z)$ introduced above by the following rule: $\Lambda_i(za) \mapsto Y_{i,aq^{-i+1}} Y_{i-1,aq^{-i+2}}^{-1}$, where $Y_0 = Y_n = 1$.

One gains a lot of insight into the structure of finite-dimensional representations of quantum affine algebras by analyzing the q–characters. For instance, it was conjectured in [FR3] and proved in [FM1] that the image of the q–character homomorphism is equal to the intersection of the kernels of certain *screening operators*, which come from the hamiltonian interpretation of the q–character homomorphism as the q–Miura transformation. This enabled us to give an algorithm for the computation of the q–characters of the fundamental representations of $U_q(\widehat{\mathfrak{g}})$ [FM1].

H. Nakajima [N] has interpreted the q–characters in terms of the cohomologies of certain quiver varieties. Using this interpretation, he was able to describe the multiplicities of irreducible representations of $U_q(\widehat{\mathfrak{g}})$ inside tensor products of the fundamental representations.

9. Deformed $\mathcal{W}$–algebras

As we mentioned above, the q–character homomorphism expresses the eigenvalues of the transfer-matrices in spin models obtained via the Bethe Ansatz (see [FR2]). The transfer-matrices form a commutative algebra in a quantum object, the quantized enveloping algebra $U_q(\widehat{\mathfrak{g}})$. But now we know that the algebra of transfer-matrices carries a Poisson structure and that the q–character homomorphism defines a Poisson map, i.e., the q–Miura transformation. This immediately raises the question as to whether the algebra of transfer-matrices (already a quantum, albeit commutative, algebra) may be further quantized. This "second quantization" was defined in [SKAO, FF5, AKOS, FR2], and it leads us to deformations of the $\mathcal{W}$–algebras.

The deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ is a two-parameter family of associative algebras. It becomes commutative in the limit $t \to 1$, where it coincides with the q–deformed classical $\mathcal{W}$–algebra discussed above. In another limit, when $t = q^\beta$ and $q \to 1$, one obtains the conformal $\mathcal{W}$–algebras which first appeared in conformal field theory (see [FF4]).

For $\mathfrak{g} = \mathfrak{sl}_2$ we have the deformed Virasoro algebra $\mathcal{W}_{q,t}(\mathfrak{sl}_2)$. It has generators $T_n, n \in \mathbb{Z}$, satisfying the relations

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z) = (q-q^{-1})(t-t^{-1})\left(\delta\left(\frac{w}{zq^2t^2}\right) - \delta\left(\frac{wq^2t^2}{z}\right)\right)$$

where $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$ and

$$f(z) = \frac{1}{1-z}\frac{(zq^2;q^4t^4)_\infty (zt^2;q^4t^4)_\infty}{(zq^4t^2;q^4t^4)_\infty (zq^2t^4;q^4t^4)_\infty}, \qquad (a;b)_\infty = \prod_{n=0}^{\infty}(1-ab^n).$$

There is also an analogue of the Miura transformation (free field realization) given by the formula

$$T(z) = :\Lambda(z): + :\Lambda(zq^2t^2)^{-1}: ,$$

where $\Lambda(z)$ is the exponential of a generating function of generators of a Heisenberg algebra.

Just like the Virasoro and other conformal $\mathcal{W}$–algebras, which are symmetries of CFT, the deformed $\mathcal{W}$–algebras appear as dynamical symmetry algebras of various models of statistical mechanics (see [LP, AJMP]).

References

[AJMP] Y. Asai, M. Jimbo, T. Miwa, Ya. Pugai, *Bosonization of vertex operators for the $A^{(1)}_{n-1}$ face model*, J. Phys. **A29** (1996) 6595–6616.

[AKOS] H. Awata, H. Kubo, S. Odake, J. Shiraishi, *Quantum $\mathcal{W}_N$ algebras and Macdonald polynomials*, Comm. Math. Phys. **179** (1996) 401–416.

[AOS] H. Awata, S. Odake, J. Shiraishi, *Free boson realization of $U_q(\widehat{\mathrm{sl}_N})$*, Comm. Math. Phys. **162** (1994) 61–83.

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint, available at www.math.uchicago.edu/∼benzvi.

[DS] V. Drinfeld and V. Sokolov, *Lie algebras and KdV type equations*, J. Sov. Math. **30** (1985) 1975–2036.

[FF1] B. Feigin, E. Frenkel, *A family of representations of affine Lie algebras*, Russ. Math. Surv. **43**, N 5 (1988) 221–222.

[FF2] B. Feigin, E. Frenkel, *Affine Kac-Moody Algebras and semi-infinite flag manifolds*, Comm. Math. Phys. **128**, 161–189 (1990).

[FF3] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras*, Int. Jour. Mod. Phys. **A7**, Supplement 1A (1992) 197–215.

[FF4] B. Feigin, E. Frenkel, *Integrals of motion and quantum groups*, in Proceedings of the C.I.M.E. School *Integrable Systems and Quantum Groups*, Italy, June 1993, Lect. Notes in Math. **1620**, pp. 349–418, Springer, 1995.

[FF5] B. Feigin, E. Frenkel, *Quantum $\mathcal{W}$–algebras and elliptic algebras*, Comm. Math. Phys. **178** (1996) 653–678.

[FFR] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe ansatz and critical level*, Comm. Math. Phys. **166** (1994) 27–62.

[F1] E. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, in Proceedings of the International Congress of Mathematical Physics, Paris, 1994, ed. D. Iagolnitzer, pp. 606–642, International Press, 1995.

[F2] E. Frenkel, *Deformations of the KdV hierarchies and related soliton equations*, Int. Math. Res. Notices. **2**, 55–76 (1996)

[F3] E. Frenkel, *Lectures on Wakimoto modules, opers and the center at the critical level*, Preprint math.QA/0210029.

[FB] E. Frenkel, D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88. AMS 2001.

[FM1] E. Frenkel, E. Mukhin, *Combinatorics of q-characters of finite-dimensional representations of quantum affine algebras*, Comm. Math. Phys. **216** (2001) 23–57.

[FM2] E. Frenkel, E. Mukhin, *The q-characters at roots of unity*, Adv. Math. **171** (2002) 139–167.

[FR1] E. Frenkel, N. Reshetikhin, *Quantum affine algebras and deformations of the Virasoro and $\mathcal{W}$–algebras*, Comm. Math. Phys. **178**, 237–264 (1996).

[FR2] E. Frenkel, N. Reshetikhin, *Deformations of $\mathcal{W}$–algebras associated to simple Lie algebras*, Comm. Math. Phys. **197** (1998), no. 1, 1–32.

[FR3] E. Frenkel, N. Reshetikhin, *The q–characters of representations of quantum affine agebras and deformations of $\mathcal{W}$–algebras*, Preprint math.QA/9810055; in Contemporary Math **248**, 163–205, AMS 2000.

[FRS] E. Frenkel, N. Reshetikhin, M.A. Semenov-Tian-Shansky, *Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$–algebras I*, Comm. Math. Phys. **192** (1998) 605–629.

[K] V.G. Kac, *Infinite-dimensional Lie Algebras*, 3rd Edition, Cambridge University Press, 1990.

[LP] S. Lukyanov, Ya. Pugai, *Multi-point local height probabilities in the integrable RSOS model*, Nuclear Phys. **B473** (1996) 631–658.

[N] H. Nakajima, *Quiver varieties and finite-dimensional representations of quantum affine algebras*, J. Amer. Math. Soc. **14** (2001) 145–238

[RS] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky, *Central extensions of quantum current groups*, Lett. Math. Phys. **19** (1990) 133–142.

[SS] M.A. Semenov-Tian-Shansky, A.V. Sevostyanov, *Drinfeld-Sokolov reduction for difference operators and deformations of $\mathcal{W}$-algebras. II. The general semisimple case*, Comm. Math. Phys. **192** (1998) 631–647.

[SKAO] J. Shiraishi, H. Kubo, H. Awata, S. Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996) 33-51.

[W] M. Wakimoto, *Fock representations of affine Lie algebra $A_1^{(1)}$*, Comm. Math. Phys. **104** (1986) 605–609.

Department of Mathematics, University of California, Berkeley, CA 94720, USA