

Definability in characteristic 0 henselian valued fields via the leading terms

Joseph Flenner

University of California, Berkeley

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- ▶ If K is a field and Γ an ordered abelian group, a valuation is a homomorphism $v : K^\times \rightarrow \Gamma$ satisfying $v(x+y) \geq \min\{v(x), v(y)\}$. This is generally extended to make $v(0) = \infty$.
- ▶ The valuation ring is $R := \{x \in K \mid v(x) \geq 0\}$. R is a local ring with maximal ideal $\mathfrak{m} := \{x \in R \mid v(x) > 0\}$.
- ▶ The residue field of K is $k := R/\mathfrak{m}$.
- ▶ We assume throughout that $\text{char}(K) = 0$.

Valued fields have a topology defined by balls.

Definition

A *ball* is a subset of K of the form K , \emptyset , $B_{\geq\gamma}(\alpha)$, or $B_{>\gamma}(\alpha) = \{x \in K \mid v(x - \alpha) > \gamma\}$.

Definition

A *swiss cheese* is a subset of K of the form $B \setminus (C_1 \cup \dots \cup C_n)$, where B, C_1, \dots, C_n are all balls (or K itself), with $C_i \subsetneq B$.

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Definition

K is *henselian* if it satisfies Hensel's Lemma:

For every $f(x) \in R[x]$ and $a \in R$, if $v(f(a)) > 0$ and $v(f'(a)) = 0$, then there exists $b \in R$ such that $\bar{b} = \bar{a}$ and $f(b) = 0$.

Every valued field has a *henselization*, an extension field which is henselian and which embeds into any other henselian extension.

The henselization of K carries the same value group Γ and residue field k as K .

In particular, no assumptions can be made about the value group or residue field of an arbitrary henselian field.

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Model theoretically, there are various languages in which one may talk about valued fields. For example, a three-sorted language K, k, Γ . But instead we will use a language based on leading term structures.

Definition

The *leading term structure* RV of K is $K^\times / (1 + \mathfrak{m})$.

The *leading term* of $x \in K^\times$ is the image $\text{rv}(x)$ of x under the quotient map.

One may also define higher order structures RV_δ for $0 \leq \delta \in \Gamma$ as $K^\times / (1 + \mathfrak{m}_\delta)$, where $\mathfrak{m}_\delta = \{x \in K \mid v(x) > \delta\}$ (so $\text{RV} = \text{RV}_0$). These will only be needed in the mixed characteristic case.

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Example

In the valued field $\mathbb{C}((t))$, two elements will have the same leading term of order n if their value is the same and their first $n + 1$ coefficients coincide:

Let

$$\begin{aligned}x &= t^{-2} + 2t^{-1} + 1 + t + 3t^2 + t^3 + t^4 + \dots \\y &= t^{-2} + 2t^{-1} + 1 + t + t^2\end{aligned}$$

Then $\text{rv}_0(x) = \text{rv}_0(y)$, $\text{rv}_3(x) = \text{rv}_3(y)$, but $\text{rv}_4(x) \neq \text{rv}_4(y)$.

Addition in RV

Besides the multiplication, RV_δ also inherits addition from K , but this addition is only partially defined. Define a relation:

$$\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \iff$$

$$\exists x, y, z \in K (\mathbf{x} = \text{rv}_\delta(x) \wedge \mathbf{y} = \text{rv}_\delta(y) \wedge \mathbf{z} = \text{rv}_\delta(z) \wedge x + y = z)$$

Proposition

Given any $x, y \in K$, $v(x + y) = \min\{v(x), v(y)\}$ iff for all z such that $\text{rv}(z) = \text{rv}(x)$, $\text{rv}(z + y) = \text{rv}(x + y)$.

We generally write $\mathbf{x} + \mathbf{y} = \mathbf{z}$ rather than $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$, but the proposition shows that the \mathbf{z} is only uniquely defined if $v(x + y) = \min\{v(x), v(y)\}$.

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Fact

For all nonzero $x, y \in K$, the following are equivalent:

1. $\text{rv}(x) = \text{rv}(y)$
2. $v(x - y) > v(x)$
3. $v(x) = v(y)$ **and** $\text{res}(y/x) = 1$

It follows that Γ and k are both interpretable in RV. Moreover, if $\varepsilon \leq \delta$, then RV_ε is interpretable in RV_δ .

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However, the two-sorted RV language is somewhat weaker than the usual three-sorted one.

The key complication arises where the addition in RV_δ is not well-defined.

As noted earlier, $\text{rv}_\delta(P(x))$ is a well-defined function of $\text{rv}_\delta(x)$ only when $v(P(x)) = \min\{v(a_i x^i)\}$ (for $P(x) = \sum_{i=0}^d a_i x^i$).

This motivates the definition

Definition

$P(x)$ has a *collision at β around α* if, for $P(x) = \sum_{i=0}^d a_i (x - \alpha)^i$,
 $v(P(\beta)) > \min_{i \leq d} \{v(a_i (\beta - \alpha)^i)\}$.

Notice that this idea of collision is not intrinsic to the polynomial alone, but depends on how P is expanded as a sum of monomials.

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Notice that this idea of collision is not intrinsic to the polynomial alone, but depends on how P is expanded as a sum of monomials.

In residue characteristic 0, collisions can only occur near a root of one of the (nonzero) derivatives of P .

Proposition

Let $\text{char}(k) = 0$, and suppose $P(x) = \sum_{i=0}^d a_i(x - \alpha)^i$ has a collision at β around α . Then there are $n < d$ and $\lambda \in K$ with

- (i) $P^{(n)}(\lambda) = 0$, and
- (ii) $\text{rv}(\lambda - \alpha) = \text{rv}(\beta - \alpha)$, and in particular,
 $v(\lambda - \beta) > v(\beta - \alpha)$.

Proof.

- ▶ Let m be maximal such that
$$\min_{i \leq d} \{v(a_i(\beta - \alpha)^i)\} = v(a_m(\beta - \alpha)^m)$$
- ▶ Define $\sigma := a_m(\beta - \alpha)^m$ and $Q(x) := \frac{1}{\sigma}P((\beta - \alpha)x + \alpha)$
- ▶ So $Q \in R[x]$, and $v(Q(1)) > 0$ (since $Q(1) = \frac{1}{\sigma}P(\beta)$ and $v(P(\beta)) > v(a_m(x - \alpha)^m) = v(\sigma)$ by definition of collision).

We will attempt to find a root of a derivative of Q using Hensel's Lemma.

Direct computation of valuations shows that

$$v(Q^{(m)}(1)) = v\left(\frac{1}{\sigma} \sum_{i=m}^d \frac{i!}{(i-m)!} a_i (\beta - \alpha)^i 1^{i-m}\right) = v(m!) = 0$$

So, let $n < m$ be least with $v(Q^{(n+1)}(1)) = 0$.

Apply Hensel's Lemma to $Q^{(n)}(x)$ to find a root $u \in K$ of $Q^{(n)}$ with $\bar{u} = \bar{1}$.

The desired root of $P^{(n)}(x)$ is $\lambda := u(\beta - \alpha) + \alpha$. □

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In residue characteristic p , this proof breaks down because $v(m!) \neq 0$ if $m \geq p$.

However, it can be shown that if Hensel's Lemma cannot give us a root of Q , then the 'severity' of the collision is bounded:

Proposition

Assume $\text{char}(k) = p > 0$. If $P(x)$ has a collision at β around α , then either there exists a root λ of a derivative of P as in the previous Proposition, or there is an integer $n \geq 0$ such that $\text{rv}(\tilde{\beta} - \alpha) = \text{rv}(\beta - \alpha)$ implies

$$\min_{i \leq d} \left\{ v(a_i(\tilde{\beta} - \alpha)^i) \right\} < v(P(\tilde{\beta})) \leq \min_{i \leq d} \left\{ v(a_i(\tilde{\beta} - \alpha)^i) \right\} + v(n)$$

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The next result uses this to decompose K into swiss cheeses on each of which $v(P(x))$ is simple, being a function of $v(x - \alpha)$ for some α . We again assume $\text{char}(k) = 0$.

Proposition

Given $P(x) \in K[x]$, there exist (disjoint) swiss cheeses V_1, \dots, V_n partitioning K , and $\alpha_1, \dots, \alpha_n \in K$ such that for all $x \in V_i$,

$$v(P(x)) = \min_{j \leq d} \{v(a_{ij}(x - \alpha_i)^j)\},$$

with $P(x) = \sum_{j=0}^d a_{ij}(x - \alpha_i)^j$.

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Vaguely:

The proof proceeds by seeking out a collision, finding a root λ of a derivative $P^{(n)}$ near them, and recentering P as $\sum b_i(x - \lambda)^i$ there.

Since there are only finitely many such λ , this process must stop, at which point there can be no further collisions nearby.

In particular, the α_i in the proposition can be chosen from among the roots of derivatives $P^{(n)}$.

Finally, translating this into the language of leading terms gives

Proposition

Assume $\text{char}(k) = 0$. Let $P(x) = \sum_{i=0}^d a_i(x - \alpha)^i \in K[x]$. Then there are disjoint swiss cheeses V_1, \dots, V_n partitioning K , $\alpha_1, \dots, \alpha_m \in K$ and terms $t_i[x_1, \dots, x_m]$ in the RV language such that for all $x \in V_i$,

$$t_i[\text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_m)]$$

is well-defined and equal to $\text{rv}(P(x))$.

The proposition provides a method of pushing questions about the field into the RV structures.

Two applications illustrate how definability in characteristic 0 henselian fields is controlled by the leading terms:

- ▶ decidability relative to the leading terms
- ▶ a characterization of definable subsets of K in terms of definable subsets in the leading terms

Both require *relative quantifier elimination*, namely in our theory every formula is equivalent to one with no *field-sorted* quantifiers.

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Both require *relative quantifier elimination*, namely in our theory every formula is equivalent to one with no *field-sorted* quantifiers.

Start with a formula of the form (suppressing parameters and free variables, where φ is a definable predicate in the RV language and the f_i are polynomials):

$$\exists x \in K (\varphi(\text{rv}(f_1(x)), \dots, \text{rv}(f_n(x))))$$



$$\exists y_1, \dots, y_m \in K \left(\bigwedge_i g_i(y_i) = 0 \wedge \exists x (\psi(\text{rv}(x - y_1), \dots, \text{rv}(x - y_m))) \right)$$

This can be translated into a formula of the above form using the decomposition. Now ψ is another predicate definable in RV and the g_i are polynomials with degrees $\leq \max \{\deg(f_i(x))\}$.

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Now, in the new formula, the $\exists x$ can be eliminated by replacing each $\text{rv}(x - y_i)$ with a new bound RV-variable z_i (this is the base case for an induction, which we've omitted).

If the degrees of the g_i are actually *strictly less* than the degrees of the f_i , we can proceed inductively.

Otherwise, we finally have to turn to a more involved argument of making successive approximations and as before relying on the fact that we know only finitely many things can go wrong.

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The quantifier elimination proof yields an explicit process of producing a field-quantifier-free formula, so that

- ▶ Given a decision procedure for the RV structure, we have also a decision procedure for the valued field
- ▶ Alternatively, there is a decision procedure for characteristic 0 henselian fields given an oracle for the RV structure

Note that the same proofs will work under any expansion of the RV sorts.

This is especially useful because we may wish to

- ▶ add a cross section (to make RV bi-interpretable with $k \times \Gamma$)
- ▶ expand RV to RV^{eq} (to work towards a relative elimination of imaginaries)

Holly proved that if K is algebraically closed, then every definable subset of K has a canonical presentation as a finite union of swiss cheeses.

Using the quantifier elimination and the swiss cheese decomposition, with some syntactic manipulation we can show:

Theorem

Every definable subset of a (pure) characteristic 0 henselian field K can be written as

$$\{x \in K \mid \langle \text{rv}(x - \alpha_1), \dots, \text{rv}(x - \alpha_n) \rangle \in D\}$$

where D is definable in RV^n .

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