DEFINABLE SUBSETS OF HENSELIAN VALUED FIELDS

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ABSTRACT. Let K be a henselian valued field of characteristic 0. Then K has a definable partition on each piece of which the leading term of a polynomial in one variable can be computed as a definable function of the leading term of a linear map. As a consequence, every definable subset of K in one dimension can be presented as a pullback of a definable set in the leading terms subjected to a translation.

1. Introduction

In [4], Holly showed that definable subsets of algebraically closed valued fields can be expressed in a canonical way as disjoint unions of *swiss cheeses*, sets of the form

$$S \setminus (T_1 \cup \ldots \cup T_n)$$

where the S, T_i are open or closed balls. In this way she presents the balls as the basic building blocks of the definable subsets of the field K. The language of valued fields used here is a three-sorted one, with sorts for the field, the value group, and the residue field.

We aim to prove a generalization of Holly's theorem for henselian valued fields of characteristic 0, and to do so adopt a language built instead around the leading term structures. These both capture the information of the value group and residue field, and provide an algebraic view of the topology of balls. Since this language also permits a strong relative quantifier elimination due to Kuhlmann [6], it seems to be a more natural setting for the theory.

The results are necessarily relativized to the leading term structures, in the sense that a simple characterization of definable subsets of the field is given in terms of definable subsets of the leading terms. This is unavoidable as, unlike in the algebraically closed case where the residue field is strongly minimal and the value group o-minimal, henselian valued fields in general make no restrictions on definability in residue field or value group. Since the leading term structure interprets the residue field and value group, the same holds true of definable sets there.

As with the work of Holly, it is hoped that this may form the one-dimensional case for an approach to a relative elimination of imaginaries in the spirit of Haskell, Hrushovski, and Macpherson [3].

The core of the proof is in Section 3, in which it is shown that the field admits a partition on each piece of which the leading term of a polynomial in x can be easily computed in terms of the leading term of $x - a$, some $a \in K$. In Section 2, the basic properties of the leading term structures are studied in detail. Section 4

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justifies the citation of Kuhlmann's quantifier elimination, and Section 5 gives the promised generalization of Holly's theorem.

2. Leading terms

2.1. **Definitions and notation.** Throughout, K will be a henselian valued field of characteristic 0, although everything in this section also applies to valued fields in general. K has value group V and valuation ring $\mathcal{O}$, with ideals

$$\mathfrak{m}_\delta := \{x \in \mathcal{O} \mid v(x) > \delta\}$$

and in particular maximal ideal $\mathfrak{m} := \mathfrak{m}_0$. The residue field then is denoted $R := \mathcal{O}/\mathfrak{m}$, and the residue of x is written either $\bar{x}$ or $\operatorname{res}(x)$ as convenient.

Valued fields possess a topology having as basic open sets the open balls

$$B_{>\delta}(a) := \{x \in K \mid v(x-a) > \delta\}$$

(note that the union of two intersecting closed balls is itself a closed ball). Closed balls $B_{\geq\delta}(a)$ are defined in the obvious way, and we will also have occasion to refer to balls of the form

$$B_{>\delta/n}(a) := \{x \in K \mid nv(x-a) > \delta\}$$

even if δ is not divisible by n in V.

Definition 2.1. Let $\delta \geq 0$ in V. The *leading term structure of order* δ is the quotient group

$$\mathrm{RV}_\delta := K^\times/(1+\mathfrak{m}_\delta).$$

The quotient map is denoted $\mathrm{rv}_\delta : K^\times \to \mathrm{RV}_\delta$. As with the value group, it is convenient to include an element ∞ in RV_δ as $\mathrm{rv}_\delta(0)$. Generally, the subscript 0 will be omitted, so $\mathrm{RV} = \mathrm{RV}_0$ and $\mathrm{rv} = \mathrm{rv}_0$.

Besides the induced multiplication, RV_δ inherits a partially defined addition via the relation

$$\oplus_\delta(\mathbf{x},\mathbf{y},\mathbf{z}) \iff \exists x,y,z \in K\,(\mathbf{x} = \mathrm{rv}_\delta(x) \wedge \mathbf{y} = \mathrm{rv}_\delta(y) \wedge \mathbf{z} = \mathrm{rv}_\delta(z) \wedge x+y=z)\,.$$

The sum $\mathbf{x}+\mathbf{y}$ is said to be *well-defined* (and $= \mathbf{z}$) if there is exactly one $\mathbf{z}$ such that $\oplus_\delta(\mathbf{x},\mathbf{y},\mathbf{z})$. The notation $\mathbf{x}+\mathbf{y} = \mathbf{z}$ will often be used for simplicity, bearing in mind that $\mathbf{x}+\mathbf{y}=\mathbf{z}$ and $\mathbf{x}+\mathbf{y}=\mathbf{w}$ does not always imply $\mathbf{z}=\mathbf{w}$. If $P(x) = \sum a_i x^i \in K[x]$, then by $P(\mathrm{rv}_\delta(x))$ we mean $\sum \mathrm{rv}_\delta(a_i)\,\mathrm{rv}_\delta(x^i)$.

If $\gamma \geq \delta \geq 0$, since $1+\mathfrak{m}_\gamma \subseteq 1+\mathfrak{m}_\delta$ there is a natural map $\mathrm{RV}_\gamma \to \mathrm{RV}_\delta$, which we also denote rv_δ, or $\mathrm{rv}_{\gamma\to\delta}$ should there be fear of confusion.

To be clear, then, the *leading term language* refers to a multisorted language

$$(K, \langle \mathrm{RV}_\delta \rangle_{\delta\in\Delta})$$

with the usual ring language on the field sort, $\Delta \subseteq \{\delta \in V \mid 0 \leq \delta < \infty\}$ to be specified as needed, the multiplication and the relation $\oplus_\delta$ on each RV_δ, and as maps between the sorts $\mathrm{rv}_\delta : K \to \mathrm{RV}_\delta$ and $\mathrm{rv}_{\gamma\to\delta} : \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$ for each $\gamma \geq \delta \in \Delta$.

The following propositions justify some of the claims of the Introduction. The proofs follow directly from the definitions.

Proposition 2.2. *For all nonzero $x, y \in K$, the following are equivalent:*

(1) $\mathrm{rv}(x) = \mathrm{rv}(y)$
(2) $v(x-y) > v(y)$
(3) $v(x) = v(y)$ *and* $\operatorname{res}(y/x) = 1$ □

Proposition 2.3. *Given $x, y \in K$ nonzero and $\delta \geq 0$ in V, the following are equivalent:*

(1) $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$
(2) $v(x-y) > v(y) + \delta$
(3) $B_{>v(x)+\delta}(x) = B_{>v(y)+\delta}(y)$ □

In particular, note that because $v(x-y) > v(y)$ can occur only when $v(x) = v(y)$, $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ implies $v(x) = v(y)$. So we can speak unambiguously of $v(\mathbf{x})$ for $\mathbf{x} \in \mathrm{RV}_\delta$ (any $\delta \geq 0$).

Next we establish when the addition on RV_δ is well-defined.

Proposition 2.4. *Let $\delta \geq 0$, and $v(x+y) = \min\{v(x), v(y)\}$. Then for all z such that $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$, $\mathrm{rv}_\delta(z+y) = \mathrm{rv}_\delta(x+y)$.*

Conversely, if $v(x+y) > \min\{v(x), v(y)\} = v(x)$, then there exists z such that $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$ but $\mathrm{rv}_\delta(z+y) \neq \mathrm{rv}_\delta(x+y)$.

Proof. Consider $z = x(1+m)$, with $v(m) > \delta$. Defining $m' := \frac{xm}{x+y}$, we then find

$$z + y = x(1+m) + y = x + y + (x+y)m' = (x+y)(1+m')$$

and

$$v(m') = v(x) + v(m) - v(x+y) \geq v(m) > \delta.$$

On the other hand, suppose $v(x+y) - v(x) = \varepsilon > 0$, and let m be any element of value $\delta + \varepsilon$. Take $z := x(1+m)$. As $v(m) > \delta$, $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$. But

$$v((z+y) - (x+y)) = v(z-x) = \delta + \varepsilon$$

implies, by Proposition 2.3, that $\mathrm{rv}_\delta(z+y) \neq \mathrm{rv}_\delta(x+y)$. □

Therefore, there is a well-defined $\mathbf{z} \in \mathrm{RV}_\delta$ such that $\oplus_\delta(\mathrm{rv}_\delta(x), \mathrm{rv}_\delta(y), \mathbf{z})$ precisely when $v(x+y) = \min\{v(x), v(y)\}$, namely $\mathbf{z} = \mathrm{rv}_\delta(x+y)$.

For later use, it will be necessary to extend 2.4 to polynomials in RV_δ. This is not entirely automatic, since even if say $v(x+y+z) = \min\{v(x), v(y), v(z)\}$, it may be the case that $\mathrm{rv}_\delta(y) + \mathrm{rv}_\delta(z)$ is not well-defined. It must then be shown that if $\oplus_\delta(\mathrm{rv}_\delta(y), \mathrm{rv}_\delta(z), \mathbf{u}_1)$ and $\oplus_\delta(\mathrm{rv}_\delta(y), \mathrm{rv}_\delta(z), \mathbf{u}_2)$ with $\mathbf{u}_1 \neq \mathbf{u}_2$, we still have $\mathrm{rv}_\delta(x) + \mathbf{u}_1 = \mathrm{rv}_\delta(x) + \mathbf{u}_2$. This however is easily accomplished with an application of Proposition 2.3.

Proposition 2.5. *Suppose $v(x_1 + \ldots + x_n) = \min\{v(x_1), \ldots, v(x_n)\}$. Then $\mathbf{y} = \mathrm{rv}_\delta(x_1) + \ldots + \mathrm{rv}_\delta(x_n)$ if and only if $\mathbf{y} = \mathrm{rv}_\delta(x_1 + \ldots + x_n)$.* □

The next proposition clarifies what happens when the addition is not well-defined.

Proposition 2.6. *Suppose that $v(x_1 + \ldots + x_n) - \min\{v(x_i)\} = \varepsilon > 0$. If $\gamma \geq \delta + \varepsilon$ and $\mathrm{rv}_\gamma(x_1) + \ldots + \mathrm{rv}_\gamma(x_n) = \mathbf{z} \in \mathrm{RV}_\gamma$, then $\mathrm{rv}_{\gamma\to\delta}(\mathbf{z}) = \mathrm{rv}_\delta(x_1 + \ldots + x_n)$.*

Proof. First, it is easy to check that $v(\mathbf{z}) = v(x_1 + \ldots + x_n) = \min\{v(x_i)\} + \varepsilon$. By definition of $\oplus_\gamma$, there are $z \in K$ and $m_i \in \mathfrak{m}_\gamma$ such that $z = x_1(1+m_1) + \ldots + x_n(1+m_n)$. Now

$$v(x_1 + \ldots + x_n - z) = v(x_1 m_1 + \ldots + x_n m_n) \geq \min\{v(x_i m_i)\}$$
$$> \min\{v(x_i)\} + \gamma \geq \min\{v(x_i)\} + \varepsilon + \delta = v(z) + \delta$$

and 2.3 give $\mathrm{rv}_\delta(x_1 + \ldots + x_n) = \mathrm{rv}_\delta(z) = \mathrm{rv}_{\gamma\to\delta}(\mathbf{z})$. □

In other words, when $v(x+y) > v(x)$, while 2.4 shows that there is more than one $\mathbf{z} \in \mathrm{RV}_\gamma$ such that $\oplus_\gamma(\mathrm{rv}_\gamma(x), \mathrm{rv}_\gamma(y), \mathbf{z})$, 2.6 implies that all such $\mathbf{z}$ have the same image in RV_δ for $\delta \leq \gamma - v(x+y) + v(x)$.

As a corollary, the following proposition shows that when $v(x+y)$ is not too much larger than $v(x)$ (compared to γ), at least $v(\mathrm{rv}_\gamma(x) + \mathrm{rv}_\gamma(y))$ is well-defined. On the other hand, when $v(x+y) > v(x) + \gamma$, nothing further can be said.

Proposition 2.7. *Suppose* $\varepsilon = v(x+y) - v(x) \geq 0$. *Then*

(i) if $\gamma \geq \varepsilon$ *and* $\oplus_\gamma(\mathrm{rv}_\gamma(x), \mathrm{rv}_\gamma(y), \mathbf{z}_1)$ *and* $\oplus_\gamma(\mathrm{rv}_\gamma(x), \mathrm{rv}_\gamma(y), \mathbf{z}_2)$*, then* $v(\mathbf{z}_1) = v(\mathbf{z}_2)$.

(ii) if $0 \leq \gamma < \varepsilon$ *and* $v(z) > v(x) + \gamma$*, then* $\oplus_\gamma(\mathrm{rv}_\gamma(x), \mathrm{rv}_\gamma(y), \mathrm{rv}_\gamma(z))$.

Proof. The first statement is 2.6 with $\delta = 0$, while the second follows from $\mathrm{rv}_\gamma(x) = \mathrm{rv}_\gamma(x+z)$, $\mathrm{rv}_\gamma(y) = \mathrm{rv}_\gamma(-x)$. □

The following example provides a good general source of intuition.

Example 2.8. Let R be any field, and V any ordered abelian group. The *Hahn field* $R((t^V))$ consists of the formal power series over R

$$\sum_{\delta \in V} c_\delta t^\delta$$

where the support $\{\delta \mid c_\delta \neq 0\}$ is well-ordered. Taking $v(\sum c_\delta t^\delta) = \min\{\delta \mid c_\delta \neq 0\}$, $R((t^V))$ has residue field R and value group V. Moreover, Hahn fields are complete with respect to the valuation, and thus henselian.

More concretely, in case $R = \mathbb{C}$ and $V = \mathbb{Z}$, we have the field $\mathbb{C}((t))$ of Laurent series over the complex numbers. Two such series will have the same leading term of order 3, say, if their first four coefficients coincide. Thus, if

$$\begin{aligned} x &= t^{-2} + t^{-1} + 1 + t + 3t^2 \\ y &= t^{-2} + t^{-1} + 1 + t + t^2 \end{aligned}$$

then $\mathrm{rv}_3(x) = \mathrm{rv}_3(y)$ since $v(x-y) = v(2t^2) = 2 > v(y) + 3$. But $\mathrm{rv}_4(x) \neq \mathrm{rv}_4(y)$.

2.2. **Interpretations.** Recall that a structure N is *interpretable* in M when there is a $\emptyset$-definable subset $S \subseteq M$ and a $\emptyset$-definable equivalence relation $\sim$ on S such that

(i) the elements of N are in bijection with the equivalence classes of $\sim$, and
(ii) the relations on S induced by the relations and functions of N by this bijection are all $\emptyset$-definable.

As suggested in 2.2, the leading term structures in a sense encompass both residue field and value group. This can now be made more explicit.

Proposition 2.9. *Things are interpretable in the leading term structures:*

(1) The value group V *is interpretable in* RV.
(2) The residue field R *is interpretable in* RV.
(3) For $\gamma > \delta \geq 0$ *in* V, RV_δ *is interpretable in* RV_γ *as long as we may use as a parameter an element* $\mathbf{d} \in \mathrm{RV}_\gamma$ *of value* $v(\mathbf{d}) = \delta$.

Proof. (1): To begin, observe that $v(\mathbf{x}) > 0$ is definable in RV. Indeed,

$$v(\mathbf{x}) > 0 \iff \mathbf{x} + \mathrm{rv}(1) = \mathrm{rv}(1).$$

From this it follows that $v(\mathbf{x}) = 0$ is also definable:

$$v(\mathbf{x}) = 0 \iff \neg v(\mathbf{x}) > 0 \wedge \exists \mathbf{y}\, (\mathbf{x}\mathbf{y} = \mathrm{rv}(1) \wedge \neg v(\mathbf{y}) > 0)\,.$$

Now define the equivalence relation $\sim$ on RV by

$$\mathbf{x} \sim \mathbf{y} \iff \exists \mathbf{c}\, (v(\mathbf{c}) = 0 \wedge \mathbf{x} = \mathbf{c}\mathbf{y})\,.$$

Clearly, we have $\mathbf{x} \sim \mathbf{y}$ iff $v(\mathbf{x}) = v(\mathbf{y})$, so that the equivalence classes of $\sim$ in RV are in bijection with V.

Moreover, addition of $v(\mathbf{x}) + v(\mathbf{y})$ in V corresponds to the multiplication $\mathbf{x}\mathbf{y}$ in RV, and the group ordering $<$ is defined by $\mathbf{x} < \mathbf{y}$ iff $\mathbf{x} \neq \infty \wedge \mathbf{x} + \mathbf{y} = \mathbf{x}$.

(2): Nonzero elements of the residue field are in bijection with $\mathbf{x} \in \mathrm{RV}$ such that $v(\mathbf{x}) = 0$, with $\bar{x}$ corresponding to $\mathrm{rv}(x)$. To see this, simply note that when $v(x) = v(y) = 0$,

$$\bar{x} = \bar{y} \iff v(x - y) \in \mathfrak{m} \iff \mathrm{rv}(x) = \mathrm{rv}(y).$$

Defining the multiplication and addition is routine.

(3): Considering $\mathbf{x} = \mathrm{rv}_\gamma(x), \mathbf{y} = \mathrm{rv}_\gamma(y) \in \mathrm{RV}_\gamma$, it will be enough to show that $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ is definable in RV_γ. Recalling 2.7, this follows from

$$\begin{aligned} \mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y) \quad &\iff \quad v(x - y) > v(y) + \delta \\ &\iff \quad \forall \mathbf{z} \in \mathrm{RV}_\gamma\, (\oplus_\gamma(\mathbf{x}, -\mathbf{y}, \mathbf{z}) \to v(\mathbf{z}) > v(\mathbf{y}) + \delta)\,. \end{aligned}$$

□

3. Decomposition

3.1. **Collisions.** The goal being to investigate definability in K through the leading term structures, this would be trivial if we could simply say that $\mathrm{rv}(P(x)) = P(\mathrm{rv}(x))$. However, as seen in 2.4, this is not always the case. For example, $\mathrm{rv}(x^2 + a) = \mathrm{rv}(x)^2 + \mathrm{rv}(a)$ only when the sum is well-defined. Problems arise where x^2 and a 'collide' to make $v(x^2 + a) > \min\{v(x^2), v(a)\}$.

Our strategy is to partition K into swiss cheeses so that on each piece of the partition, $v(P(x))$ reduces to a simple expression, and $\mathrm{rv}_\delta(P(x))$ can be analyzed within RV_δ in linear terms.

Definition 3.1. $P(x) = \sum_{i=0}^{d} a_i (x - \alpha)^i$ has a *collision at* β *around* α if $v(P(\beta)) > \min_{i \le d}\{v(a_i(\beta - \alpha)^i)\}$.

As mentioned above, for any x where $P(x)$ does *not* have a collision,

$$\mathrm{rv}_\delta(P(x)) = \sum_{i=0}^{d} \mathrm{rv}_\delta(a_i)\, \mathrm{rv}_\delta(x - \alpha)^i$$

is well-defined. Clearly, having a collision at β around α depends only on $\mathrm{rv}(\beta - \alpha)$. The next proposition shows that there are finitely many leading terms where a collision can happen.

Proposition 3.2. *Let* $\alpha \in K$ *and* $P(x) = \sum_{i=0}^{d} a_i (x - \alpha)^i$. *There are finitely many leading terms* $\mathrm{rv}(\beta - \alpha)$ *for which* P *has a collision at* β *around* α.

Proof. There are at most $d(d+1)/2$ values $\delta \in V$ for which P has a collision at β with $v(\beta-\alpha)=\delta$. This is because there can only be a collision if $v(a_i(\beta-\alpha)^i) = v(a_j(\beta-\alpha)^j)$ for some $0 \le i < j \le d$, and this equation has at most one solution in V.

So, fixing a $\delta \in V$, we show that there are finitely many leading terms $\mathbf{u}$ such that $v(\mathbf{u}) = \delta$ and there is a collision at some β with $\mathrm{rv}(\beta-\alpha) = \mathbf{u}$. Pick any β with $v(\beta-\alpha)=\delta$, let $k \le d$ be such that $v(a_k(\beta-\alpha)^k) = v(a_k)+k\delta = \min\limits_{i\le d}\{v(a_i)+i\delta\}$, and define

$$Q(x) := \frac{1}{a_k(\beta-\alpha)^k} P((\beta-\alpha)x+\alpha) = \frac{1}{a_k(\beta-\alpha)^k}\sum_{i=0}^{d} a_i(\beta-\alpha)^i x^i.$$

$Q(x) \in \mathcal{O}[x]$, and the residue polynomial $\overline{Q}$ is nonzero. If P has a collision at $\tilde\beta$ around α, and $v(\tilde\beta-\alpha)=\delta$, then setting $u := \frac{\tilde\beta-\alpha}{\beta-\alpha}$ we have $v(u)=0$ and

$$v\left(Q(u)\right) = v\left(\sum_{i=0}^{d} a_i(\tilde\beta-\alpha)^i\right) - \min_{i\le d}\left\{v\left(a_i(\beta-\alpha)^i\right)\right\} > 0.$$

So $\bar u$ is a root of $\overline{Q}$ in R.

Furthermore, if $\mathrm{rv}(\tilde\beta-\alpha) \ne \mathrm{rv}(\beta'-\alpha)$ and $w := \frac{\beta'-\alpha}{\beta-\alpha}$, then $v(\tilde\beta-\beta)=\delta$ gives

$$v(u-w) = v\left(\frac{\tilde\beta-\beta'}{\beta-\alpha}\right) = 0.$$

So $\bar u \ne \bar w$, and $\tilde\beta$ and β' give rise to different roots of $\overline{Q}$. The conclusion follows. $\square$

In residue characteristic 0, we can go further by locating collisions near roots of the derivatives of P. Here let us introduce the convention that if $\deg(P)=d$ then by *derivatives of* P we mean P, P', ..., and $P^{(d)}$, notably including P itself as the 0^{th} derivative.

Proposition 3.3. *Assume* $\mathrm{char}(R)=0$. *Suppose* $P(x) = \sum\limits_{i=0}^{d} a_i(x-\alpha)^i$ *has a collision at* β *around* α. *Then there are* $n<d$ *and* $\lambda \in K$ *with*

(i) $P^{(n)}(\lambda)=0$, *and*

(ii) $\mathrm{rv}(\lambda-\alpha) = \mathrm{rv}(\beta-\alpha)$, *and in particular,* $v(\lambda-\beta) > v(\beta-\alpha)$.

Proof. First note that $\beta \ne \alpha$, as otherwise the inequality in Definition 3.1 could not be satisfied. Let m be maximal such that $\min_{i\le d}\{v(a_i(\beta-\alpha)^i)\} = v(a_m(\beta-\alpha)^m)$, and as in 3.2 define $\sigma := a_m(\beta-\alpha)^m$ and $Q(x) := \frac{1}{\sigma}P((\beta-\alpha)x+\alpha)$. So $Q \in \mathcal{O}[x]$, and $v(Q(1))>0$.

Consider $Q^{(m)}(1)$. Since

$$Q^{(m)}(x) = \frac{1}{\sigma}\sum_{i=m}^{d}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i x^{i-m},$$

for $i=m$ we have

$$v\left(\frac{1}{\sigma}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i 1^{i-m}\right) = v\left(\frac{m!}{\sigma}a_m(\beta-\alpha)^m\right) = v(m!) = 0 \tag{1}$$

while for $i > m$

$$(2) \qquad v\left(\frac{1}{\sigma}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i 1^{i-m}\right) = v(a_i(\beta-\alpha)^i) - v(a_m(\beta-\alpha)^m) > 0$$

by the maximality of m. (Note that here we are using $v(n) = 0$ for all $n \in \mathbb{Z}$, a consequence of $\mathrm{char}(R) = 0$.) Thus $v(Q^{(m)}(1)) = 0$.

Now let n be least such that $v(Q^{(n+1)}(1)) = 0$. The above shows that n is at most $m-1$. Applying Hensel's Lemma to $Q^{(n)}$, there is a $u \in \mathcal{O}$ with $\bar{u} = \bar{1} \in R$ and $Q^{(n)}(u) = 0$. It follows that $v(u-1) > 0 = v(1)$, so that $\mathrm{rv}(u) = \mathrm{rv}(1)$.

Let $\lambda := u(\beta-\alpha)+\alpha$. So $\mathrm{rv}(\lambda-\alpha) = \mathrm{rv}(u)\,\mathrm{rv}(\beta-\alpha) = \mathrm{rv}(\beta-\alpha)$. This implies that $v(\lambda-\beta) = v((\lambda-\alpha)-(\beta-\alpha)) > v(\beta-\alpha)$ as in (ii). Finally,

$$0 = Q^{(n)}(u) = \frac{(\beta-\alpha)^n}{\sigma}P^{(n)}((\beta-\alpha)u+\alpha) = \frac{(\beta-\alpha)^n}{\sigma}P^{(n)}(\lambda).$$

Since $\alpha \neq \beta$, $P^{(n)}(\lambda) = 0$. □

The situation when $\mathrm{char}(R) = p$ is more complicated. Comparing the calculations in (1) and (2) above, we find $v(Q^{(m)}(1)) = v(m!)$ (note that $v(i!/(i-m)!) \geq v(m!)$, since $m!$ divides $i!/(i-m)!$). If $v(Q(1)) > 2^m v(m!)$, Hensel's Lemma would still apply for whichever $v(Q^{(i)}(1)) > 2v(Q^{(i+1)}(1))$, and so we could find a root λ as before.

Otherwise, the same argument will work for any $\tilde{\beta}$ such that $\mathrm{rv}(\tilde{\beta}-\alpha) = \mathrm{rv}(\beta-\alpha)$ and

$$\widetilde{Q}(x) = \frac{1}{a_m(\tilde{\beta}-\alpha)^m}P((\tilde{\beta}-\alpha)x+\alpha).$$

So if there is no root λ of a derivative of P, it must be that $\widetilde{Q}(1) \leq 2^m v(m!)$ for every $\tilde{\beta} \in B_{>v(\beta-\alpha)}(\beta)$. Therefore

$$v(P(\tilde{\beta})) = v(a_m(\tilde{\beta}-\alpha)^m) + v(\widetilde{Q}(1)) \leq \min_{i \leq d}\left\{v(a_i(\tilde{\beta}-\alpha)^i)\right\} + v((m!)^{2^m})$$

and we have proven

Proposition 3.4. *Assume* $\mathrm{char}(R) = p > 0$. *If $P(x)$ has a collision at β around α, then either there exists a root λ of a derivative of P as in Proposition 3.3, or there is an integer $q > 0$ such that* $\mathrm{rv}(x-\alpha) = \mathrm{rv}(\beta-\alpha)$ *implies*

$$\min_{i \leq d}\left\{v(a_i(x-\alpha)^i)\right\} < v(P(x)) \leq \min_{i \leq d}\left\{v(a_i(x-\alpha)^i)\right\} + v(q).$$

Moreover, q can be chosen no larger than $(d!)^{2^d}$. □

3.2. **The decomposition.** Like the m in the proof of Proposition 3.3, we will frequently need to refer to the largest degree term carrying the smallest valuation. Therefore define

$$(3) \quad m(P,\alpha,S) := \max\left\{i \leq d \mid \exists x \in S\ \forall j \leq d\ \left(v(a_i(x-\alpha)^i) \leq v(a_j(x-\alpha)^j)\right)\right\}$$

where the a_i are the coefficients of the expansion of P around α,

$$P(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i.$$

Thus, $m(P,\alpha,S)$ is the highest order term in P centered at α which can have minimal valuation (among the other terms of P) on S.

As before, we first prove the main result for residue characteristic 0, and then indicate the modifications needed if $\mathrm{char}(R) = p$.

Proposition 3.5. *Suppose* $\mathrm{char}(R) = 0$. *Let* $P(x) \in K[x]$ *and* S *be a swiss cheese in* K. *Then there exist (disjoint) sub-swiss cheeses* $T_1, \ldots, T_k \subseteq S$ *and* $\alpha_1, \ldots, \alpha_k \in K$, *all algebraic over the coefficients of* P *and parameters defining* S, *such that*

$$S = \dot{\bigcup} T_i$$

and for all $x \in T_i$,

$$v(P(x)) = v(a_{im_i}(x - \alpha_i)^{m_i}),$$

with $P(x) = \sum_{n=0}^{d} a_{in}(x - \alpha_i)^n$ *and* $m_i = m(P, \alpha_i, T_i)$.

Proof. For simplicity, assume S is a ball $B_{\geq\gamma}(\alpha)$. No generality is lost as a decomposition for $B_{\geq\gamma}(\alpha) \supseteq S$ may simply be intersected with S to get the desired result.

Let $P(x) = \sum_{n=0}^{d} a_i(x - \alpha)^i$. The proof proceeds by induction on $m(P, \alpha, S)$. Clearly, if $m(P, \alpha, S) = 0$, then $v(P(x)) = v(a_0)$ for all $x \in S$.

Now suppose $m(P, \alpha, S) = m$. Let

$$D := \{\delta \geq \gamma \mid \forall i \leq m \, (v(a_m) + m\delta \leq v(a_i) + i\delta)\} .$$

Intuitively, $m(P, \alpha, S) = m$ when $v(a_m(x - \alpha)^m)$ is minimal *somewhere* in S, while D gives those values where it actually *is* minimal.

D is an initial segment of $[\gamma, \infty)$. Indeed, if $\gamma \leq \varepsilon < \delta \in D$ and $i < m$, then $v(a_i) + i\delta \geq v(a_m) + m\delta$ implies

$$v(a_i) + i\varepsilon > v(a_m) + m\varepsilon \tag{4}$$

so $\varepsilon \in D$. We need not consider $i > m$, by the maximality of m.

In particular, the inequality in (4) becomes strict for $\varepsilon < \delta$. Therefore we have also shown that if $\delta \in D$ is not a maximal element of D, then for all x such that $v(x - \alpha) = \delta$,

$$v(P(x)) = v(a_m(x - \alpha)^m). \tag{5}$$

Suppose first that D has no maximal element, and define

$$B_D := \{x \in S \mid v(x - \alpha) \in D\} .$$

Then there is some $i < m$ such that $v(a_m(x - \alpha)^m) > v(a_i(x - \alpha)^i)$ whenever $v(x - \alpha) > \delta$ for every $\delta \in D$. Set $\eta := v(a_i) - v(a_m)$ and note that

$$S \setminus B_D = B_{\geq\gamma}(\alpha) \setminus B_D = B_{>\eta/(m-i)}(\alpha)^1$$

since for $x \in S$,

$$\begin{aligned} x \notin B_D &\Leftrightarrow v(a_i(x - \alpha)^i) < v(a_m(x - \alpha)^m) \\ &\Leftrightarrow v(a_i) - v(a_m) < (m - i)v(x - \alpha). \end{aligned}$$

Therefore, if $x \in B_D$ then $v(P(x)) = v(a_m(x - \alpha)^m)$ by (5), whereas if $x \in S \setminus B_D$ then $m(P, \alpha, B_{>\eta/(m-i)}(\alpha)) < m$ and the induction hypothesis applies.

[1] It should be pointed out here that the failure of D to have a maximal element means that η is not divisible by $m - i$ in V, but as noted earlier we may still define the ball $B_{>\eta/(m-i)}(\alpha)$.

On the other hand, suppose $D = [\gamma, \delta]$ has a maximum at δ. Now S decomposes into three swiss cheeses:

$$S \;=\; B_{>\delta}(\alpha) \;\cup\; B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) \;\cup\; B_{\geq\gamma}(\alpha) \setminus B_{\geq\delta}(\alpha).$$

On the last of these, as observed above, $v(P(x)) = v(a_m(x-\alpha)^m)$. On the first, $m(P, \alpha, B_{>\delta}(\alpha)) < m$, so that again, the induction hypothesis applies. It therefore remains only to consider $B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) =: A$, i.e. when $v(x-\alpha) = \delta$.

Let $C := \{x \in A \mid v(P(x)) \neq v(a_m(x-\alpha)^m)\}$, so C is the set of elements of A at which P has a collision around α. Now, C is the disjoint union of equivalence classes under the equivalence $x \sim y \Leftrightarrow v(x-y) > \delta \Leftrightarrow \mathrm{rv}(x-\alpha) = \mathrm{rv}(y-\alpha)$.

Proposition 3.2 shows that there are finitely many such equivalence classes. Furthermore, by Proposition 3.3, each of the equivalence classes contains a root λ of a derivative of P.

So, C is a finite union of balls of the form $B_{>\delta}(\lambda)$, having centers algebraic over the coefficients of P. Thus we see that $A \setminus C$ is a swiss cheese on which $v(P(x)) = v(a_m(x-\alpha)^m)$, so now it remains only to determine $v(P(x))$ on C.

Choose a $\lambda \in C$, $P^{(n)}(\lambda) = 0$, and let

$$P(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i = \sum_{i=0}^{d} b_i(x-\lambda)^i.$$

Taking $\sigma := a_m(\lambda-\alpha)^m$, define once more

$$Q_\alpha(x) := \frac{1}{\sigma} P((\lambda-\alpha)x+\alpha) = \frac{1}{\sigma}\sum_{i=0}^{d} a_i(\lambda-\alpha)^i x^i$$

$$Q_\lambda(x) := \frac{1}{\sigma} P((\lambda-\alpha)x+\lambda) = \frac{1}{\sigma}\sum_{i=0}^{d} b_i(\lambda-\alpha)^i x^i$$

Note that $Q_\alpha(x+1) = Q_\lambda(x)$ and that $Q_\alpha(x), Q_\lambda(x) \in \mathcal{O}[x]$. However, since

$$v(a_i(\lambda-\alpha)^i) > v(a_m(\lambda-\alpha)^m)$$

for $m < i \leq d$, the residue polynomial $\overline{Q_\alpha}(x)$ has degree m. Now it follows from $Q_\alpha(x+1) = Q_\lambda(x)$ that $\overline{Q_\lambda}(x)$ also has degree m.

By considering the residues of the coefficients of Q_λ, then, we get $v(b_i(\lambda-\alpha)^i) > v(\sigma)$ for $i > m$ and $v(b_m(\lambda-\alpha)^m) = v(\sigma)$. Since $v(\lambda-\alpha) = \delta$, in shifting from $B_{\geq\gamma}(\alpha)$ to $B_{>\delta}(\lambda)$, we see that

$$m(P, \alpha, B_{\geq\gamma}(\alpha)) \geq m(P, \lambda, B_{>\delta}(\lambda)).$$

But equality may occur, so that we cannot yet invoke the induction. Instead, if indeed $m(P, \lambda, B_{>\delta}(\lambda)) = m$, the proof concludes with a second induction on the number k of roots of the nonconstant derivatives of P contained in the ball $B_{>\delta}(\lambda)$.

As above, we decompose $B_{>\delta}(\lambda)$ into the three Swiss cheeses $B_{>\varepsilon}(\lambda)$, $B_{\geq\varepsilon}(\lambda) \setminus B_{>\varepsilon}(\lambda)$, and $B_{\geq\delta}(\lambda) \setminus B_{\geq\varepsilon}(\lambda)$; and as above, each case is dealt with easily except where collisions occur within $B_{\geq\varepsilon}(\lambda) \setminus B_{>\varepsilon}(\lambda) =: A'$.

For the base case $k = 1$ of the induction (not $k = 0$, as $B_{>\delta}(\lambda)$ contains at least the root λ), A' is empty, so we are done. Otherwise, there are strictly fewer than k roots of derivatives of P in A', again because $\lambda \notin A'$, so here the induction step gives the rest of the decomposition.

To finish, note that throughout the proof everything has been definable from the coefficients of P and parameters defining S, with the exception of the (finitely many) balls comprising C and the centers λ of those balls, which as roots of derivatives of P are algebraic over the coefficients of P. □

Now, in residue characteristic p, the proof above remains valid except when looking in C, not every ball necessarily contains a root λ. In those that don't, however, as in Proposition 3.4 we can bound by $v(q)$ the amount that the collision forces $v(P(x))$ up. So the corresponding modification reads

Proposition 3.6. *Suppose* $\operatorname{char}(R) = p$. *Let* $P(x) \in K[x]$ *and* S *be a swiss cheese in* K. *Then there exist (disjoint) sub-swiss cheeses* $T_1, \ldots, T_k \subseteq S$ *and* $\alpha_1, \ldots, \alpha_k \in K$, *all algebraic over the coefficients of* P *and parameters defining* S, *such that*

$$S = \dot{\bigcup} T_i$$

and for each T_i, *either*

$$\forall x \in T_i \left(v(P(x)) = v(a_{im_i}(x - \alpha_i)^{m_i}) \right)$$

(with $P(x) = \sum_{n=0}^{d} a_{in}(x-\alpha_i)^n$ *and* $m_i = m(P, \alpha_i, T_i)$*); or there is an integer* $q \in \mathbb{N}$, $0 < q \leq (d!)^{2^d}$, *such that*

$$\forall x \in T_i \left(\min_{j \leq d} \left\{ v(a_{ij}(x - \alpha_i)^j) \right\} < v(P(x)) \leq \min_{j \leq d} \left\{ v(a_{ij}(x - \alpha_i)^j) \right\} + v(q) \right).$$

□

Finally, we return to the leading term structures to find expressions for analyzing $\operatorname{rv}_\delta(P(x))$. Thanks to Propositions 2.5 and 2.6, this is an immediate consequence of the above two propositions.

Proposition 3.7. *Let* $P(x) = \sum_{i=0}^{d} a_i x^i \in K[x]$ *and* $0 \leq \delta \in V$. *Then there are disjoint swiss cheeses* $U_1, \ldots, U_k$ *partitioning* $K = \bigcup_{i=1}^{k} U_i$ *and* $\alpha_1, \ldots, \alpha_k \in K$ *such that for each* i, *if* $P(x) = \sum_{j=0}^{d} a_{ij}(x - \alpha_i)^j$ *then for all* $x \in U_i$ *either*

(i) $\operatorname{rv}_\delta(P(x)) = \sum_{j=0}^{d} \operatorname{rv}_\delta(a_{ij}) \operatorname{rv}_\delta(x - \alpha_i)^j$ *is well-defined, or*

(ii) there is a positive integer $q \leq (d!)^{2^d}$ *such that*

$$\operatorname{rv}_\delta(P(x)) = \operatorname{rv}_\delta\left(\sum_{j=0}^{d} \operatorname{rv}_{\delta+v(q)}(a_{ij}) \operatorname{rv}_{\delta+v(q)}(x - \alpha_i)^j \right)$$

is well-defined.

The latter case only occurs in positive residue characteristic.

Furthermore, the $\alpha_1, \ldots, \alpha_k$ *and* $U_1, \ldots, U_k$ *can be chosen to be algebraic over* $\{a_0, \ldots, a_d\}$. □

As a final note, though each of the preceding propositions is stated for a single polynomial P, the same results will hold for any finite number of polynomials $P_1, \ldots, P_n$. To obtain the desired decomposition, simply apply the proposition to each P_i separately, and then intersect the resulting partitions to get one which works for all P_i simultaneously.

The methods used in the decomposition given above are reminiscent of those employed by Cohen [2] in his decision procedure for the p-adics. In fact, it can be shown that these results and techniques can be used to give an effective quantifier elimination, and therefore a decision procedure, for the field relative to the leading term structures. Details to appear in future work.

One may also compare the cell decomposition for characteristic 0 henselian fields given in the context of b-minimality. See [1].

4. Quantifier elimination

In the following section, we use the above to give a characterization of all definable subsets of K in terms of definable subsets of the leading term structures. In order to do this, quantifier elimination relative to the leading terms is needed. This means that every formula over a henselian field of characteristic 0 is equivalent to one with no quantifiers over the field sort; equivalently, that the theory admits quantifier elimination after expanding to the morleyization of the leading term sorts.

Such a result has been proven by Kuhlmann [6] relative to the *amc-structures*. He defines, for each $\delta \geq 0$ in V, the system

$$K_\delta := (\mathcal{O}_\delta, G_\delta, \Theta_\delta(x,y))$$

whereby $\mathcal{O}_\delta := \mathcal{O}/\mathfrak{m}_\delta$, $G_\delta := K^\times/(1+\mathfrak{m}_\delta)$ $(= \mathrm{RV}_\delta$ as a multiplicative group). The notations π_δ and π^*_δ are used for the canonical maps $\mathcal{O} \to \mathcal{O}_\delta$ and $K^\times \to G_\delta$ respectively (again, π^*_δ is the same as our rv_δ). Then the relation Θ_δ is defined on $\mathcal{O}_\delta \times G_\delta$ as

$$\Theta_\delta(x,y) \Leftrightarrow \exists z \in \mathcal{O} \ \ (\pi_\delta(z) = x \wedge \pi^*_\delta(z) = y)\,.$$

Kuhlmann proves that in residue characteristic 0, there is quantifier elimination relative to K_0, while in residue characteristic p it is relative to the family $\{K_\delta \ \mid \ \delta = v(p^n), n \in \mathbb{N}\}$.

In fact Kuhlmann's language of amc-structures is equivalent to leading terms. To obtain the needed relative quantifier elimination it will suffice to show this.

Proposition 4.1. *For $\delta \geq 0$, K_δ is interpretable in RV_δ.*

Proof. Since G_δ is a retract of RV_δ, it only remains to give an interpretation of $\mathcal{O}_\delta$, as well as of the relation Θ_δ.

Elements $a, b \in \mathcal{O}$ have equal images in $\mathcal{O}_\delta$ if $a - b \in \mathfrak{m}_\delta$, that is, if $v(a-b) > \delta$. Using the notation

$$\mathrm{RV}^+_\delta := \mathrm{RV}_\delta(\mathcal{O}) = \{\mathbf{x} \in \mathrm{RV}_\delta \mid 0 \leq v(\mathbf{x})\}$$

define an equivalence relation $\sim$ on RV^+_δ by

$$\mathbf{a} \sim \mathbf{b} \Longleftrightarrow \forall \mathbf{c} \in \mathrm{RV}_\delta \, (\mathbf{a} - \mathbf{b} = \mathbf{c} \rightarrow v(\mathbf{c}) > \delta)\,.$$

It is clear that there is a bijection between $\mathcal{O}_\delta$ and the equivalence classes of $\sim$ on RV^+_δ.

In fact, if we define the surjection

$$\vartheta : \mathrm{RV}^+_\delta \twoheadrightarrow \mathcal{O}_\delta$$

so that $\vartheta(\mathbf{x})$ is the element of $\mathcal{O}_\delta$ corresponding to the equivalence class of $\mathbf{x}$, then we have $\pi_\delta = \vartheta \circ \mathrm{rv}_\delta$.

It remains to show that the graphs of addition

$$A := \left\{ \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\mathrm{RV}_\delta^+)^3 \mid \vartheta(\mathbf{x}) + \vartheta(\mathbf{y}) = \vartheta(\mathbf{z}) \right\}$$

and multiplication

$$M := \left\{ \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\mathrm{RV}_\delta^+)^3 \mid \vartheta(\mathbf{x})\vartheta(\mathbf{y}) = \vartheta(\mathbf{z}) \right\}$$

as well as the interpreted form of the relation Θ_δ,

$$\widetilde{\Theta}_\delta := \left\{ \langle \mathbf{x}, \mathbf{y} \rangle \in (\mathrm{RV}_\delta^+)^2 \mid \exists z \in \mathcal{O} \ \ (\pi_\delta(z) = \vartheta(\mathbf{x}) \wedge \mathrm{rv}_\delta(z) = \mathbf{y}) \right\},$$

are definable in RV_δ.

Beginning with $\widetilde{\Theta}_\delta$, it happens in fact that

$$\widetilde{\Theta}_\delta(\mathbf{x}, \mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y},$$

i.e. iff $\vartheta(\mathbf{x}) = \vartheta(\mathbf{y})$. This is because given $y \in \mathcal{O}$ with $\mathrm{rv}_\delta(y) = \mathbf{y}$,

$$\pi_\delta(y) = \vartheta(\mathbf{x}) \Leftrightarrow \vartheta(\mathrm{rv}_\delta(y)) = \vartheta(\mathbf{y}) = \vartheta(\mathbf{x}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}.$$

Similarly, $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in M$ iff $\mathbf{xy} \sim \mathbf{z}$ follows from again taking $x, y, z \in \mathcal{O}$ with $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(y) = \mathbf{y}, \mathrm{rv}_\delta(z) = \mathbf{z}$ and noticing that

$$\pi_\delta(x)\pi_\delta(y) = \pi_\delta(z) \Leftrightarrow \pi_\delta(xy) = \pi_\delta(z) \Leftrightarrow \vartheta(\mathbf{xy}) = \vartheta(\mathbf{z}) \Leftrightarrow \mathbf{xy} \sim \mathbf{z}.$$

Finally, for addition we have $A = \oplus_\delta \cap \left(\mathrm{RV}_\delta^+\right)^3$. To see this, taking $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{RV}_\delta^+$, we have

$$\begin{aligned} \oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \quad &\Leftrightarrow \quad \exists x, y, z \in \mathcal{O}\, (\mathrm{rv}_\delta(x) = \mathbf{x} \wedge \mathrm{rv}_\delta(y) = \mathbf{y} \wedge \mathrm{rv}_\delta(z) = \mathbf{z} \wedge x + y = z) \\ &\Leftrightarrow \quad \exists x, y, z \in \mathcal{O}\, (\pi_\delta(x) + \pi_\delta(y) = \pi_\delta(z)) \\ &\Leftrightarrow \quad \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in A. \end{aligned}$$

□

Though it is not explicitly needed for our purposes, we note that the reverse is also true.

Proposition 4.2. RV_δ *is interpretable in* K_δ.

Proof. It is only necessary to prove that the relation $\oplus_\delta$ is definable on G_δ. Suppose for simplicity that $\mathbf{x} \neq \infty$ and $v(\mathbf{x}) \leq v(\mathbf{y})$. We claim that $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ if and only if

$$\exists \mathbf{x^{-1}} \in G_\delta \exists \bar{a}, \bar{b} \in \mathcal{O}_\delta \left(\mathbf{x^{-1}x} = \mathbf{1} \wedge \Theta_\delta(\bar{a}, \mathbf{x^{-1}y}) \wedge \Theta_\delta(\bar{b}, \mathbf{x^{-1}z}) \wedge \bar{1} + \bar{a} = \bar{b} \right).$$

If there are $x, y, z \in K$ with $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(y) = \mathbf{y}, \mathrm{rv}_\delta(z) = \mathbf{z}$ and $x + y = z$, then $1 + x^{-1}y = x^{-1}z$ gives $\bar{1} + \bar{a} = \bar{b}$ in the above. Conversely, if the formula holds, then taking $x^{-1}, a, b \in K$ such that $\mathrm{rv}_\delta(x^{-1}) = \mathbf{x^{-1}}$, $\pi_\delta(a) = \bar{a}$, and $\pi_\delta(b) = \bar{b}$, we find that $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(xa) = \mathbf{y}, \mathrm{rv}_\delta(xb) = \mathbf{z}$ witness $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$. □

It is evident that 4.1 and 4.2 in fact give a bi-interpretation. Consequently, the choice of the leading term language over the amc-structure language is purely stylistic. Kuhlmann's theorem now states

Proposition 4.3 (Kuhlmann [6])**.** *The theory of* $(K, \langle \mathrm{RV}_\delta \rangle_{\delta\in\Delta})$ *admits elimination of field-sorted quantifiers, where*

$$\Delta = \begin{cases} \{0\} & \text{if } \mathrm{char}(R) = 0 \\ \{v(p^n) \mid n \in \mathbf{N}\} & \text{if } \mathrm{char}(R) = p > 0. \end{cases}$$

□

5. Definable subsets of K

In this section, the goal is to use the quantifier elimination and decomposition to give a characterization of definable subsets of K. As mentioned in the introduction, this is intended as a generalization of the theorem of Holly on canonical forms for sets definable in algebraically closed valued fields (see [4]).

Proposition 5.1. *Suppose* $S \subseteq K$ *is definable over* A*. Then there are* $\alpha_1, \dots, \alpha_k \in \mathrm{acl}(A)$ *and a subset* $D \subseteq \mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_k}$ *definable over* $\mathrm{acl}(A)$ *such that*

$$S = \{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x - \alpha_1), \dots, \mathrm{rv}_{\delta_k}(x - \alpha_k)\rangle \in D\}\,.$$

As before, if $\mathrm{char}(R) = 0$*, we may take* $\delta_i = 0$ *for all* i*; if* $\mathrm{char}(R) = p > 0$*, then the* δ_i *can be taken among* $v(p^n)$ *for* $n \in \mathbb{N}$*.*

Proof. The elimination of field-sorted quantifiers from Proposition 4.3 implies that S is definable by a formula the form

$$\varphi\left(\mathrm{rv}_{\delta_1}(f_1(x)), \dots, \mathrm{rv}_{\delta_k}(f_k(x))\right) \tag{6}$$

with φ being a formula in the RV sorts and each f_i a polynomial with coefficients over A.

Applying the decomposition of Proposition 3.7, there are swiss cheeses $U_1, \dots, U_m$ partitioning K, for each $i \le k$ RV polynomials $t_{i1}, \dots, t_{im}$ (over $\mathrm{acl}(A)$), and for each $i \le k$ and $j \le m$ field elements $\alpha_{ij} \in \mathrm{acl}(A)$ such that (6) is equivalent to

$$\bigvee_{j=1}^{m} \left(x \in U_j \ \wedge \varphi\left(t_{1j}[\mathrm{rv}_{\delta_{1j}}(x - \alpha_{1j})], \dots, t_{kj}[\mathrm{rv}_{\delta_{kj}}(x - \alpha_{kj})]\right)\right)$$

(with each $\delta_{ij} = \delta_i + v(p^n)$, some $n \in \mathbb{N}$, or $\delta_{ij} = 0$ in residue characteristic 0).

The condition $x \in U_j$ is definable in RV with parameters of the form $\mathrm{rv}(x - \beta)$. Without loss of generality we consider β to be among the α_{ij} and let ψ_j be the RV formula expressing

$$\psi_j(\mathbf{x}_1, \dots, \mathbf{x}_k) \Longleftrightarrow x \in U_j \ \wedge \varphi\left(t_{1j}[\mathbf{x}_1], \dots, t_{kj}[\mathbf{x}_k]\right).$$

For each $i \le k$ define $\gamma_i := \max_{j\le m}\{\delta_{ij}\}$. Since every $t_{ij}[\mathrm{rv}_{\delta_{ij}}(x - \alpha_{ij})]$ can be computed as $t_{ij}[\mathrm{rv}_{\gamma_i \to \delta_{ij}}(\mathrm{rv}_{\gamma_i}(x - \alpha_{ij}))]$, it may without loss of generality be assumed that $\delta_{ij} = \gamma_i$ for all i, j. Thus each ψ_j is a formula over $\mathrm{RV}_{\gamma_1} \times \ldots \times \mathrm{RV}_{\gamma_k}$.

Finally, letting χ be the formula $\bigvee \psi_j$ and D be the set in $\mathrm{RV}_{\gamma_1} \times \ldots \times \mathrm{RV}_{\gamma_k}$ defined by χ, we have

$$S = \{x \in K \mid \langle \mathrm{rv}_{\gamma_1}(x - \alpha_1), \dots, \mathrm{rv}_{\gamma_k}(x - \alpha_k)\rangle \in D\}$$

as required. □

Holly's swiss cheeses in algebraically closed valued fields arise as boolean combinations of a finite number of balls. This can be seen as the combination of a pullback of a finite set (from the residue field) and an interval (the value group). It is a consequence of strong minimality and o-minimality that these are all the sets definable in residue field and value group. As pointed out in the Introduction, it is a necessary byproduct of the relativity in the henselian setting that we must allow for pullbacks of arbitrary definable sets D in the leading term structures.

The pullback of an interval in the value group itself will produce a ball (or, more accurately, an annulus) around 0. Shifting to balls centered elsewhere can be taken as analogous to our linear shifting by $\langle \alpha_1, \ldots, \alpha_k \rangle$.

To obtain a one-dimensional elimination of imaginaries in [5] ('1-prototypes'), Holly introduces a new sort for the balls. It follows by the same reasoning that henselian valued fields of characteristic 0 admit 1-prototypes in the leading term language after adding new sorts for definable sets of the form

$$\{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x - \alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x - \alpha_k) \rangle \in D\}.$$

In more dimensions, it is an immediate consequence of quantifier elimination that definable subsets of K^n take the form

$$\{\langle x_1, \ldots, x_n \rangle \in K \mid \langle \mathrm{rv}_{\delta_1}(f_1(\bar{x})), \ldots, \mathrm{rv}_{\delta_k}(f_k(\bar{x})) \rangle \in E\} \tag{7}$$

where E is definable in $\mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_k}$ and each $f_i \in K[x_1, \ldots, x_n]$.

One could then obtain a trivial elimination of imaginaries by including the sets (7) as new sorts. An approach towards a more satisfying solution of the elimination of imaginaries problem may be to give a necessary and sufficient subclass of the polynomials f_i.

For example, one could hope to show that every definable set can be coded in terms of sets of the form (7) with the f_i being affine transformations of K^n. This seems overly optimistic, but would provide a suitable analogy to Haskell, Hrushovski, and Macpherson's elimination of imaginaries for algebraically closed valued fields [3] in terms of definable modules and torsors over $\mathcal{O}$.

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