The Relative Structure of Henselian Valued Fields

by

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B.S. (University of Michigan) 2000

A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

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Fall 2008

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Fall 2008

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Abstract

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In [12], Holly showed that definable subsets of algebraically closed valued fields can be expressed in a canonical way as disjoint unions of swiss cheeses. In this way she presents the balls as the basic building blocks of the definable subsets of the field, and thereby deduces a one-dimensional elimination of imaginaries for algebraically closed valued fields.

We aim to prove a generalization of Holly's theorem for henselian valued fields of characteristic 0, and to do so adopt a language built around the leading term structures. If rv_δ is the leading term map of order δ and $f(x) = \sum_{j=0}^{d} a_j x^j$ is a polynomial, we construct a decomposition of the field $K = \dot{\bigcup_i} S_i$ such that on each S_i, if f is rewritten as $f(x) = \sum_{j=0}^{d} a_{ij}(x - \alpha_i)^j$ then

$$\mathrm{rv}_\delta(f(x)) = \mathrm{rv}_\delta\left(\sum_{j=0}^{d} \mathrm{rv}_\gamma(a_{ij})\,\mathrm{rv}_\gamma(x - \alpha_i)^j\right)$$

for some $\gamma = \delta + v(n)$ ($n \in \mathbb{Z}_+$ a positive integer).

This is then used to show that every definable subset of K (in one variable) can be expressed as a translation of a pullback of a set definable in the leading terms. From this, an expansion of the leading term language for characteristic 0 henselian valued fields is found in which one-dimensional elimination of imaginaries holds.

As with the work of Holly, it is hoped that this may form the one-dimensional case for an approach to a relative elimination of imaginaries in the spirit of Haskell, Hrushovski, and Macpherson [10].

As a byproduct of the decomposition, in residue characteristic 0 we also give a more explicit relative quantifier elimination procedure (which holds as well in an expansion of the language on the leading term sorts). This provides an effective decision procedure for pure characteristic 0 henselian valued fields relative to the leading term structure.

Professor Thomas Scanlon
Dissertation Committee Chair

To my parents, and Holly.

Contents

Acknowledgments

First of all I want to thank my advisor. His patience, generosity, and good humor have been nothing short of legendary during my unusually long tenure as his student. This thesis owes an enormous debt to him intellectually, and this thesis writer to him personally.

Another benefit that comes with being a student of Professor Scanlon is the large community of like-minded aspiring model theorists. They've all been very helpful, both with mathematical things and practical issues, especially those that were here from the beginning: John Goodrick, Maryanthe Malliaris, and Alice Medvedev.

I would like to thank Professors Deirdre Haskell and Dugald Macpherson for many helpful discussions and for their hospitality during (and after) trips to Hamilton and Leeds.

During the first three years of my graduate studies at Berkeley, I was generously supported by a National Defense Science and Engineering Grant. I have also received much-appreciated financial support to attend conferences and workshops from the American Institute of Mathematics, the Association for Symbolic Logic, research grants of Thomas Scanlon and Anand Pillay, and the International Centre for the Mathematical Sciences.

Finally, as a graduate student, I have benefited greatly from the needling of many of my fellow students, in particular Jameel Al-Aidroos, Aubrey Clayton, Luke Clossey, Amanda Colligan, Kristine de Leon, Brighten Godfrey, Joyce Kim, Grace Lyo, Zachary Mesyan, Carl Miller, and David Spivak. I have also learned a lot from discussions with my students, who are too numerous to name here. And above all, I am immeasurably indebted to my family for their support.

Chapter 0

Introduction

0.1 Background

Valued fields have enjoyed a long and fruitful relationship with model theory, dating from Abraham Robinson's 1950s theorem on the completeness of the theory of algebraically closed valued fields (ACVF) (in [29]), and the work of Ax-Kochen [1],[2],[3] and Cohen [7] in the 1960s. Ax-Kochen worked with henselian valued fields, which satisfy requirements for existence of roots of certain polynomials weaker than those of the full algebraic closure. The most common examples of henselian fields are the p-adics $\mathbb{Q}_p$ and the field of Laurent series $\mathbb{C}((t))$.

Ax-Kochen showed that the theory of a henselian field of residue characteristic 0 is completely determined by the theory of its residue field R and its value group V. This could be viewed as a version of Robinson's theorem relativized to the residue field and value group, meaning loosely that if the theories of R and V are taken as a 'black box', this suffices to specify completely the full theory of the valued field.

More recently, ACVF has served as a starting point for a more general program of applying methods of stable model theory to structures which are not stable, but may contain and in a sense be controlled by a stable part. This analysis begins with the work of Haskell, Hrushovski, and Macpherson in [10] and [11], and is pursued further in particular in Hrushovski's metastability. In the first paper, it is proved that ACVF admits elimination of imaginaries (a concept originating with Poizat [24], though Shelah introduced M^{eq} in [32]) after adding sorts for definable modules and cosets of modules ("torsors") over the valuation ring.

Their proof built substantially on ideas appearing in work by Holly. In her papers [12] and [13], Holly used Robinson's quantifier elimination for ACVF to prove a canonical form theorem for definable subsets (in one variable) of an algebraically closed valued field K, using as building blocks the balls (sets of the form $\{x \in K \mid v(x-a) > \gamma\}$ or $\{x \mid v(x-a) \geq \gamma\}$ for some $a \in K$, $\gamma \in V$) and swiss cheeses (balls with finitely many sub-balls removed).

Specifically, it is proven in [12] that every definable subset of K can be expressed canonically as a finite union of swiss cheeses. She uses this to prove partial elimination of imaginaries for ACVF in [13], only for quotients of K (not quotients of powers of K), given an extra sort for the balls. The idea was that this should serve as the base case for an induction to prove the full elimination of imaginaries. A key insight in Haskell-Hrushovski-Macpherson [10] was the identification of the torsors as the natural generalization of the balls to subsets in more than one variable.

Meanwhile, there have been some initial attempts to extend the ideas of [10] to produce elimination of imaginaries for other classes of valued fields. Mellor proved such a result for real closed valued fields in [23]. Hrushovski and Martin [16] did the same for the p-adics. Their proof is not native to $\mathbb{Q}_p$, but uses a means of pulling down the elimination of imaginaries from the algebraic closure. A similar tactic of applying results from the elimination of imaginaries in the algebraic closure is also used in Hrushovski and Kazhdan's work on motivic integration [15].

0.2 Relativization

In the more general case of henselian valued fields, much of the focus until the 1990s had remained on p-adic fields. Besides Ax-Kochen and Cohen, Macintyre [21] proved an elimination of quantifiers for p-adically closed fields in a language featuring the n^{th}-power predicates $P_n := \{x \in K \mid \exists y\, (y^n = x)\}$. This suggests the dependence of definable sets in the field on definability in the residue field and value group: in $\mathbb{Q}_p$, the residue field is finite while the value group, whose theory is Presburger arithmetic

$\mathrm{Th}(\mathbb{Z}, +, <, 0)$, admits elimination of quantifiers granted divisibility predicates.

ACVF itself gives another example, with the residue field of an algebraically closed valued field being algebraically closed (and so strongly minimal), and the value group being divisible (hence o-minimal). The stable domination in ACVF gives a firm picture on how much of the model-theoretic niceness of algebraically closed valued fields is inherited from the model-theoretic niceness of R and V.

In arbitrary henselian valued fields, the picture is complicated by the fact that henselian fields may carry any field and any ordered abelian group as residue field and value group. This gives rise to the necessity for general results on henselian valued fields to be relativized. This means, for example, that while definable sets in a henselian field may be very complicated, they can be described simply in terms of the definable sets in the residue field and value group.

However, here the language of residue field and value group does not seem to be quite the right one. Instead, results have been found using various manifestations of the language of leading terms. Leading terms, under various different guises, have featured prominently in the model theory of valued fields in work of Basarab and Kuhlmann [4],[5],[20], Cluckers and Loeser [6], Hrushovski and Kazhdan [15], and Scanlon [30], among others.

In this dissertation, we seek to unify this viewpoint and lay the foundation for a deeper analysis of henselian valued fields (of characteristic 0) relativized to the leading term structures, along the lines of Haskell, Hrushovski, and Macpherson for ACVF. We prove relative henselian versions of the theorems of Holly [12],[13] on definable subsets in one dimension, and of Cohen [7] on decidability.

The technical core of the work is in Chapter 2, where a means of analyzing the leading term of a polynomial as a function of the leading term of a linear factor is given. This leads to the possibility of pushing questions about the field into the leading term structure modulo a linear shift. Thus it is found, in Chapter 4, that definable sets in one dimension of a characteristic 0 henselian valued field are translations of pullbacks of definable sets in the leading term structure.

Chapter 1

Valued Fields and Leading Terms

In the first two sections, we go through the basic definitions, properties, and examples of valued fields that will be needed. All the results here are standard. The third section introduces a different structure associated to a valued field, the *leading term structures*, which will form the basis for the language in the following chapters. For more general reading on valuation theory, we refer to [31]; in a less general context, [27]; and for valued fields specifically, [8].

1.1 Valued fields

Definition 1.1.1. A *valued field* (K, v) is a field together with a *valuation* map

$$v : K \twoheadrightarrow V \cup \{\infty\}$$

from K onto an ordered abelian group V (the *value group*) extended to $V \cup \{\infty\}$ in the natural way, such that

(i) $v(x) = \infty$ iff $x = 0$

(ii) $v(xy) = v(x) + v(y)$ (i.e. v is a group homomorphism $K^\times \to V$ of the multiplicative group of the field)

(iii) $v(x + y) \geq \min\{v(x), v(y)\}$ (the so-called *ultrametric inequality*).

(Note: Some authors prefer to write V multiplicatively, in which case the ordering is generally reversed and $v(0) = 0$.)

Valuations are a modification of the idea of an *absolute value*, which are maps from a field $|\ | : K \to \mathbb{R}_{\geq 0}$ respecting multiplication and satisfying the triangle inequality $|x + y| \leq |x| + |y|$. If in fact $|n| \leq 1$ for all $n \in \mathbb{Z}$, the absolute value is said to be *non-archimedean.* It can be shown (see [27], Chapter 1) that a non-archimedean absolute value satisfies $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$, a strengthening of the triangle inequality similar to the ultrametric inequality (with the ordering reversed). Thus it is apparent that valuations are a generalization of non-archimedean absolute values to arbitrary value groups.

Example 1.1.2. Suppose $p \in \mathbb{Z}$ is a prime. Define a map on the rational numbers by

$$v_p : p^n \frac{a}{b} \mapsto n$$

when $\gcd(a, p) = \gcd(b, p) = 1$. Then $(\mathbb{Q}, v_p)$ is a valued field, and v_p is called the *p-adic valuation.* In this case, $V = \mathbb{Z}$ under addition.

The construction of Example 1.1.2 could clearly be applied to any unique factorization domain U and prime $p \in U$, giving a p-adic valuation on the fraction field of U. For example, the function field $\mathbb{Q}(t)$.

In fact, $\mathbb{Q}(t)$ carries another very natural valuation based on the degree of polynomials. However, in associating a polynomial with its degree, while

$$\deg(P(t)Q(t)) = \deg\left(P(t)\right) + \deg(Q(t))\,,$$

the sum $P(t)+Q(t)$ has degree *at most* the maximum of $\deg(P)$ and $\deg(Q)$. The inequality operates in the opposite direction from 1.1.1(iii). This is rectified by reversing the ordering.

Example 1.1.3. Let k be a field, and $k(t)$ the field of rational functions over k. Then the map

$$v : \frac{P(t)}{Q(t)} \mapsto \deg(Q) - \deg(P)$$

defines a valuation on $k(t)$.

Notice that for $\deg(P+Q)$ to be *strictly* less than $\max\{\deg(P), \deg(Q)\}$, it is necessary (but not sufficient) that $\deg(P) = \deg(Q)$. Thus this example also illustrates a first simple principle.

Proposition 1.1.4. *Let (K, v) be a valued field. For all $x \in K$, $v(x) = v(-x)$. If $v(x) \neq v(y)$, then $v(x + y) = \min\{v(x), v(y)\}$.*

Proof. Clearly $v(1) = 0$, so $0 = v((-1)(-1)) = v(-1) + v(-1)$. But V is an ordered group, whence $v(-1) = -v(-1)$ implies $v(-1) = 0$. The first statement follows.

Now, say $v(x) < v(y)$. This gives

$$v(x) = v(x + y - y) \geq \min\{v(x + y), v(-y)\} \geq \min\left\{\min\{v(x), v(y)\}, v(y)\right\} = v(x).$$

Since $v(y) > v(x)$, we must have $\min\{v(x + y), v(y)\} = v(x + y) = v(x)$. $\square$

Definition 1.1.5. If (K, v) is a valued field, then the *valuation ring* of K is

$$\mathcal{O} := \{x \in K \mid v(x) \geq 0\}.$$

It is easily verified that $\mathcal{O}$ is indeed a ring.

In the following, a *terminal segment* of $V \cup \{\infty\}$ refers to a set C satisfying

$$\delta \in C \text{ and } \gamma \geq \delta \Longrightarrow \gamma \in C.$$

Proposition 1.1.6. *(a) Let $\mathfrak{i}$ be an ideal of $\mathcal{O}$. Then*

$$\mathfrak{i} = \{x \in \mathcal{O} \mid v(x) \in C\}$$

for some nonempty terminal segment $C \subseteq V \cup \{\infty\}$.

(b) $\mathcal{O}$ is a local ring with unique maximal ideal

$$\mathfrak{m} = \{x \in \mathcal{O} \mid v(x) > 0\}.$$

Proof. (a) If $a \in \mathfrak{i}$ is nonzero and $v(b) \geq v(a)$, then $b/a \in \mathcal{O}$ shows that $b \in \mathfrak{i}$.

(b) This is a simple consequence of (a).

$\square$

Definition 1.1.7. The *residue field* of K is the residue class field $R := \mathcal{O}/\mathfrak{m}$. The residue of $x \in \mathcal{O}$ will be denoted either $\bar{x}$ or $\operatorname{res}(x)$ as convenient.

It is easily seen that if $\operatorname{char}(R) = 0$, then $\operatorname{char}(K) = 0$ as well, whereas if $\operatorname{char}(K) = p > 0$, then $\operatorname{char}(R) = p$. However, it may occur that $\operatorname{char}(K) = 0$ while $\operatorname{char}(R) = p$. This is the case in Example 1.1.2 (where $\mathcal{O}$ is $\mathbb{Z}$ localized at the principal ideal (p), and $R = \mathbb{F}_p$ is the finite field with p elements). In Example 1.1.3, $\operatorname{char}(K) = \operatorname{char}(R)$ as $R = k$.

If $L \supseteq K$ is a valued field extension of K, the extension is called *immediate* if K and L have both the same value group and the same residue field.

Example 1.1.8. If k is a field, let $k((t))$ be the field of Laurent series over k with the valuation

$$v : \sum_{i=n}^{\infty} a_i t_i \mapsto n$$

if $a_n \neq 0$.

Then $k((t))$ is an immediate extension of the valued field $k(t)$ from 1.1.3. However, it was shown by Krull [19] that $k((t))$ is *maximal*, in the sense that it has no nontrivial immediate extension.

In fact, Krull showed this for a larger class of fields of formal power series.

Example 1.1.9. Let R be any field, and V any ordered abelian group. The *Hahn field* $R((t^V))$ consists of the formal power series over R:

$$\sum_{\delta \in V} a_\delta t^\delta$$

where the support $\{\delta \mid a_\delta \neq 0\}$ is well-ordered.

Taking

$$v\left(\sum_{\delta \in V} a_\delta t^\delta\right) = \min\left\{\delta \mid a_\delta \neq 0\right\},$$

$\big(R((t^V)), v\big)$ is a valued field with residue field R and value group V. Moreover, Hahn fields are maximal.

See Kaplansky [17], [18] for a detailed analysis of maximal valued fields, and in particular when a maximal valued field can be expressed in the form of a Hahn field. Poonen [25] has a similar construction producing maximal fields in mixed characteristic.

1.2 The valuation topology

Much like a norm, a valuation gives a concept of distance on its field. In fact, from a real-valued valuation it is possible to construct a metric on K from v as follows.

Proposition 1.2.1. *Given a valued field* (K, v) *with value group* $V \subseteq \mathbb{R}$*, define a function*

$$d : K \times K \mapsto \mathbb{R}$$

by $d(x, y) = e^{-v(x-y)}$ *(and setting* $e^{-\infty} = 0$*). This* d *metrizes* K*.*

The role of e here is simply as a convenient base for the exponent. Any real number greater than 1 will work equally well.

Proof. Symmetry comes from 1.1.4. For the triangle inequality, we have for all $x, y, z \in K$,

$$\begin{aligned} d(x,y) + d(y,z) &\geq \max\{d(x,y), d(y,z)\} = e^{-\min\{v(x-y),v(y-z)\}} \\ &\geq e^{-v((x-y)+(y-z))} = e^{-v(x-z)} = d(x,z). \end{aligned}$$

The first inequality comes from positivity, the second from the ultrametric inequality. □

Even when the valuation is not real-valued, a topology can be defined on K by analogy with the metric space topology. The distance between a and b in K is determined by $v(a-b)$, however we will say that a is close to b if $v(a-b)$ is large. Therefore it could be said that $v(a-b)$ measures the proximity of a and b.

Now K adopts a topology with basic open sets the *open balls*,

$$B_{>\delta}(\alpha) := \{x \in K \mid v(x-a) > \delta\}$$

(of radius δ). We define the closed balls $B_{\geq\delta}(\alpha)$ in the same way with $\geq$ in place of $>$, and will also have occasion to refer to balls of the form

$$B_{\geq\delta/n}(\alpha) := \{x \in K \mid nv(x-a) > \delta\}$$

even if δ is not divisible by n in V. K and $\emptyset$ are considered both open and closed balls.

The terms 'open' and 'closed' distinguish the strictness of the inequality, but in fact topologically the open balls are also closed sets, and vice versa. For example $B_{\geq\delta}(\alpha) = \bigcup B_{>\delta}(\beta)$, the union taken over all $\beta \in B_{\geq\delta}(\alpha)$ (see Proposition 1.2.2).

The topology is Hausdorff, but otherwise has some initially counterintuitive features as a consequence of the strengthening of the triangle inequality to the ultrametric inequality.

Proposition 1.2.2. *(a) If $\beta \in B_{>\delta}(\alpha)$, then $B_{>\delta}(\alpha) = B_{>\delta}(\beta)$.*

(b) Suppose that A and B are balls in K. If $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$.

Proof. (a) Consider $x \in B_{>\delta}(\beta)$. So, $v(x-\beta) > \delta$, and $v(\beta-\alpha) > \delta$ gives

$$v(x-\alpha) \geq \min\{v(x-\beta), v(\beta-\alpha)\} > \delta.$$

Thus $x \in B_{>\delta}(\alpha)$ and $B_{>\delta}(\beta) \subseteq B_{>\delta}(\alpha)$. Equality follows by symmetry, since $\alpha \in B_{>\delta}(\beta)$.

(b) Suppose for simplicity that A and B are open balls, $A = B_{>\delta}(\alpha)$ and $B = B_{>\gamma}(\beta)$ with $\gamma \geq \delta$. If $x \in A \cap B$, then by (a), $A = B_{>\delta}(x)$ and $B = B_{>\gamma}(x)$. Now $B \subseteq A$ follows from $\gamma \geq \delta$.

□

Accordingly, though it is sometimes intuitively useful to imagine the valuation topology on K by analogy with euclidean space, it can also be misleading. A more faithful picture of the topology gives K a locally treelike structure. See Holly [14] for details.

Example 1.2.3. The p-adic numbers $\mathbb{Q}_p$ can be constructed from $\mathbb{Q}$ with the p-adic valuation (see 1.1.2) by metric space completion. Alternatively, they can also be built algebraically as the fraction field of the p-adic integers

$$\mathbb{Z}_p = \varprojlim \left(\mathbb{Z}/p^n\mathbb{Z}\right).$$

Like 1.1.8, $\mathbb{Q}_p$ is a maximal immediate extension of $(\mathbb{Q}, v_p)$.

1.3 The leading term structures

We define a series of structures associated to a valued field. These both capture the information of the value group and residue field, and provide an algebraic view of the topology of balls.

In addition to the maximal ideal $\mathfrak{m}$ introduced in 1.1.6, the ideals

$$\mathfrak{m}_\delta := \{x \in \mathcal{O} \mid v(x) > \delta\}$$

for $\delta \geq 0$ are also needed. So, $\mathfrak{m} = \mathfrak{m}_0$.

Before the following definition, let us note that for each $\delta \geq 0$ in V, $1 + \mathfrak{m}_\delta$ is a subgroup of the multiplicative group $K^\times$. In particular, to see that $1 + \mathfrak{m}_\delta$ is closed under inverses, we have for $v(m) > \delta$

$$v\left((1+m)^{-1} - 1\right) = v\left(1 - (1+m)\right) - v(1+m) = v(-m) - 0 = v(m) > \delta.$$

Definition 1.3.1. Let $\delta \geq 0$ in V. The *leading term structure of order* δ is the quotient group

$$\mathrm{RV}_\delta := K^\times/(1 + \mathfrak{m}_\delta).$$

The quotient map is denoted $\mathrm{rv}_\delta : K^\times \to \mathrm{RV}_\delta$. As with the value group, it is convenient to include an element ∞ in RV_δ as $\mathrm{rv}_\delta(0)$. Generally, the subscript 0 will be omitted, so $\mathrm{RV} = \mathrm{RV}_0$ and $\mathrm{rv} = \mathrm{rv}_0$.

Besides the induced multiplication, RV_δ inherits a partially defined addition via the relation

$$\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \iff \exists x, y, z \in K \left(\mathbf{x} = \mathrm{rv}_\delta(x) \wedge \mathbf{y} = \mathrm{rv}_\delta(y) \wedge \mathbf{z} = \mathrm{rv}_\delta(z) \wedge x + y = z\right).$$

The sum $\mathbf{x} + \mathbf{y}$ is said to be *well-defined* (and $= \mathbf{z}$) if there is exactly one $\mathbf{z}$ such that $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$. The notation $\mathbf{x} + \mathbf{y} = \mathbf{z}$ will often be used for simplicity, bearing in mind that $\mathbf{x} + \mathbf{y} = \mathbf{z}$ and $\mathbf{x} + \mathbf{y} = \mathbf{w}$ does not always imply $\mathbf{z} = \mathbf{w}$. If $P(x) = \sum a_i x^i \in K[x]$, then by $P(\mathrm{rv}_\delta(x))$ we mean $\sum \mathrm{rv}_\delta(a_i)\,\mathrm{rv}_\delta(x^i)$. For this to make sense, it must be verified that the sum as defined by $\oplus_\delta$ is associative.

In fact, although the sum of more than two elements in RV_δ may not be well-defined, it is indeed associative (and commutative). It is an easy consequence of the definition of $\oplus_\delta$ and an induction on n that the sum

$$((\ldots(\mathbf{x}_1 + \mathbf{x}_2) + \ldots + \mathbf{x}_{n-1}) + \mathbf{x}_n) = \mathbf{z}$$

if and only if there are $x_i, z \in K$, $\mathrm{rv}_\delta(x_i) = \mathbf{x}_i$ and $\mathrm{rv}_\delta(z) = \mathbf{z}$, such that $x_1 + \ldots + x_n = z$.

The Hahn fields of Example 1.1.9 provide a good general source of intuition, and will also be useful later.

Example 1.3.2. In the Hahn field with $R = \mathbb{C}$ and $V = \mathbb{Z}$, we have the field $\mathbb{C}((t))$ of Laurent series over the complex numbers. Two such series will have the same leading term of order 3, say, if they have the same value and if their first four coefficients coincide (see Proposition 1.3.4). Thus, if

$$\begin{aligned} x &= t^{-2} + t^{-1} + 1 + t + 3t^2 \\ y &= t^{-2} + t^{-1} + 1 + t + t^2 \end{aligned}$$

then $v(x) = v(y) = -2$, and $\mathrm{rv}_3(x) = \mathrm{rv}_3(y)$ since $v(x-y) = v(2t^2) = 2 > v(y) + 3$. But $\mathrm{rv}_4(x) \neq \mathrm{rv}_4(y)$.

Note also that $\mathrm{rv}(x) + \mathrm{rv}(y)$ here is well-defined, while $\mathrm{rv}(x) - \mathrm{rv}(y)$ is not.

The following propositions illustrate how the leading terms encapsulate both the algebraic information of residue field and value group and the valuation topology.

Proposition 1.3.3. *For all nonzero $x, y \in K$, the following are equivalent:*

1. $\mathrm{rv}(x) = \mathrm{rv}(y)$
2. $v(x-y) > v(y)$
3. $v(x) = v(y)$ *and* $\mathrm{res}(y/x) = 1$

Proof. 1⇒2: If $\mathrm{rv}(x) = \mathrm{rv}(y)$ then there is an $m \in \mathfrak{m}$ such that $x = y(1+m)$. So $v(x-y) = v(ym) = v(y) + v(m) > v(y)$.

2⇒3: Proposition 1.1.4 gives $v(x) = v(y)$. Now, from $v(x-y) > v(y)$ we get $v(1 - y/x) > v(y) - v(x) = 0$. Thus $1 - y/x \in \mathfrak{m}$ as required.

3⇒1: Let $m := 1 - y/x$. We have $xm = x - y$ and $y = x(1-m)$. If $m \in \mathfrak{m}$, this shows that $\mathrm{rv}(x) = \mathrm{rv}(y)$.[1] □

Thus if x and y have the same leading term, they must have the same value; and if that value is 0, they must also have the same residue. A similar phenomenon occurs in the higher order structures. Indeed, if $\gamma \geq \delta \geq 0$, since $1 + \mathfrak{m}_\gamma \subseteq 1 + \mathfrak{m}_\delta$ there is a natural map $\mathrm{RV}_\gamma \longrightarrow \mathrm{RV}_\delta$, which we also denote rv_δ, or $\mathrm{rv}_{\gamma \to \delta}$ should there be fear of confusion.

Therefore, for any $\delta \geq 0$, $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ implies $v(x) = v(y)$ (and $\mathrm{res}(y/x) = 1$). So we can always speak unambiguously of $v(\mathbf{x})$ for $\mathbf{x} \in \mathrm{RV}_\delta$.

However, in RV_δ the statement can be refined somewhat.

Proposition 1.3.4. *Given $x, y \in K$ nonzero and $\delta \geq 0$ in V, the following are equivalent:*

1. $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$
2. $v(x - y) > v(y) + \delta$
3. $B_{>v(x)+\delta}(x) = B_{>v(y)+\delta}(y)$

Proof. 1⇒2: This is the same as in 1.3.3, but with $m \in \mathfrak{m}_\delta$.

2⇒1: Let $m := \frac{x-y}{y}$. Then $v(m) > \delta$ by (2), and $x = y(1+m)$ implies $\mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$.

2⇔3: Since $v(x) = v(y)$, by 1.2.2 $B_{>v(x)+\delta}(x) = B_{>v(y)+\delta}(y)$ iff $x \in B_{>v(y)+\delta}(y)$. But the latter condition is simply the statement $v(x-y) > v(y) + \delta$. □

Next we establish when the addition on RV_δ is well-defined.

Proposition 1.3.5. *Let $\delta \geq 0$, and $v(x+y) = \min\{v(x), v(y)\}$. Then for all z such that $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$, $\mathrm{rv}_\delta(z+y) = \mathrm{rv}_\delta(x+y)$.*

Conversely, if $v(x+y) > \min\{v(x), v(y)\} = v(x)$, then there exists z such that $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$ but $\mathrm{rv}_\delta(z+y) \neq \mathrm{rv}_\delta(x+y)$.

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[1] Note that $v(x) = v(y)$ is not really used here, because in fact it is implied by $\mathrm{res}(y/x) = 1$: if $v(x) > v(y)$, $y/x \notin \mathcal{O}$ so $\mathrm{res}(y/x)$ is undefined, while if $v(y) > v(x)$, $v(y/x) > 0 \Rightarrow \mathrm{res}(y/x) = 0$.

Proof. Consider $z = x(1+m)$, with $v(m) > \delta$. Defining $m' := \frac{xm}{x+y}$, we then find

$$z + y = x(1+m) + y = x + y + (x+y)m' = (x+y)(1+m')$$

and

$$v(m') = v(x) + v(m) - v(x+y) \geq v(m) > \delta.$$

On the other hand, suppose $v(x+y) - v(x) = \varepsilon > 0$, and let m be any element of value $\delta + \varepsilon$. Take $z := x(1+m)$. As $v(m) > \delta$, $\mathrm{rv}_\delta(z) = \mathrm{rv}_\delta(x)$. But

$$v((z+y) - (x+y)) = v(z-x) = \delta + \varepsilon$$

implies, by Proposition 1.3.4, that $\mathrm{rv}_\delta(z+y) \neq \mathrm{rv}_\delta(x+y)$. $\square$

Therefore, there is a well-defined $\mathbf{z} \in \mathrm{RV}_\delta$ such that $\oplus_\delta(\mathrm{rv}_\delta(x), \mathrm{rv}_\delta(y), \mathbf{z})$ precisely when $v(x+y) = \min\{v(x), v(y)\}$, namely $\mathbf{z} = \mathrm{rv}_\delta(x+y)$.

For later use, it will be necessary to extend 1.3.5 to polynomials in RV_δ. This is not entirely automatic, since even if say $v(x+y+z) = \min\{v(x), v(y), v(z)\}$, it may be the case that $\mathrm{rv}_\delta(y) + \mathrm{rv}_\delta(z)$ is not well-defined. It must then be shown that if $\oplus_\delta(\mathrm{rv}_\delta(y), \mathrm{rv}_\delta(z), \mathbf{u}_1)$ and $\oplus_\delta(\mathrm{rv}_\delta(y), \mathrm{rv}_\delta(z), \mathbf{u}_2)$ with $\mathbf{u}_1 \neq \mathbf{u}_2$, we still have $\mathrm{rv}_\delta(x) + \mathbf{u}_1 = \mathrm{rv}_\delta(x) + \mathbf{u}_2$. This however is easily accomplished with an application of Proposition 1.3.4.

Proposition 1.3.6. *Suppose* $v(x_1 + \ldots + x_n) = \min\{v(x_1), \ldots, v(x_n)\}$. *Then*

$$\mathbf{y} = \mathrm{rv}_\delta(x_1) + \ldots + \mathrm{rv}_\delta(x_n)$$

if and only if $\mathbf{y} = \mathrm{rv}_\delta(x_1 + \ldots + x_n)$.

Proof. Suppose $\mathrm{rv}_\delta(x_i) = \mathrm{rv}_\delta(y_i)$ for $i \leq n$. So there are $m_i \in \mathfrak{m}_\delta$ with $y_i = x_i(1+m_i)$. Now

$$v(x_1 + \ldots + x_n - y_1 - \ldots - y_n) = v(x_1 m_1 + \ldots + x_n m_n) > \min_{i \leq n}\{v(x_i)\} + \delta$$

implies by 1.3.4 that $\mathrm{rv}_\delta(x_1 + \ldots + x_n) = \mathrm{rv}_\delta(y_1 + \ldots + y_n)$, and the claim follows. $\square$

The next proposition clarifies what happens when the addition is not well-defined.

Proposition 1.3.7. *Suppose that* $v(x_1+\ldots+x_n)-\min\{v(x_i)\}=\varepsilon>0$. *If* $\gamma\geq\delta+\varepsilon$ *and* $\mathrm{rv}_\gamma(x_1)+\ldots+\mathrm{rv}_\gamma(x_n)=\mathbf{z}\in\mathrm{RV}_\gamma$, *then* $\mathrm{rv}_{\gamma\to\delta}(\mathbf{z})=\mathrm{rv}_\delta(x_1+\ldots+x_n)$.

Proof. First, it is easy to check that $v(\mathbf{z})=v(x_1+\ldots+x_n)=\min\{v(x_i)\}+\varepsilon$. By definition of $\oplus_\gamma$, there are $z\in K$ and $m_i\in\mathfrak{m}_\gamma$ such that $z=x_1(1+m_1)+\ldots+x_n(1+m_n)$. Now

$$v(x_1+\ldots+x_n-z)=v(x_1m_1+\ldots+x_nm_n)\geq\min\{v(x_im_i)\}$$

$$>\min\{v(x_i)\}+\gamma\geq\min\{v(x_i)\}+\varepsilon+\delta=v(z)+\delta$$

and 1.3.4 give $\mathrm{rv}_\delta(x_1+\ldots+x_n)=\mathrm{rv}_\delta(z)=\mathrm{rv}_{\gamma\to\delta}(\mathbf{z})$. □

In other words, when $v(x+y)>v(x)$, while 1.3.5 shows that there is more than one $\mathbf{z}\in\mathrm{RV}_\gamma$ such that $\oplus_\gamma(\mathrm{rv}_\gamma(x),\mathrm{rv}_\gamma(y),\mathbf{z})$, 1.3.7 implies that all such $\mathbf{z}$ have the same image in RV_δ for $\delta\leq\gamma-v(x+y)+v(x)$.

As a corollary, the following proposition shows that when $v(x+y)$ is not too much larger than $v(x)$ (compared to γ), at least $v(\mathrm{rv}_\gamma(x)+\mathrm{rv}_\gamma(y))$ is well-defined. On the other hand, when $v(x+y)>v(x)+\gamma$, nothing further can be said.

Proposition 1.3.8. *Suppose* $\varepsilon=v(x+y)-v(x)\geq 0$. *Then*

(i) if $\gamma\geq\varepsilon$ *and* $\oplus_\gamma(\mathrm{rv}_\gamma(x),\mathrm{rv}_\gamma(y),\mathbf{z}_1)$ *and* $\oplus_\gamma(\mathrm{rv}_\gamma(x),\mathrm{rv}_\gamma(y),\mathbf{z}_2)$, *then* $v(\mathbf{z}_1)=v(\mathbf{z}_2)$.

(ii) if $0\leq\gamma<\varepsilon$ *and* $v(z)>v(x)+\gamma$, *then* $\oplus_\gamma(\mathrm{rv}_\gamma(x),\mathrm{rv}_\gamma(y),\mathrm{rv}_\gamma(z))$.

Proof. The first statement is 1.3.7 with $\delta=0$, while the second follows from $\mathrm{rv}_\gamma(x)=\mathrm{rv}_\gamma(x+z)$, $\mathrm{rv}_\gamma(y)=\mathrm{rv}_\gamma(-x)$. □

1.4 Hensel's Lemma

A key subclass of valued fields is given by those satisfying

Hensel's Lemma. *If* $P(x)\in\mathcal{O}[x]$ *is a polynomial over the valuation ring with* $v(P(a))>0$ *and* $v(P'(a))=0$, *then there exists* $b\in\mathcal{O}$ *such that* $P(b)=0$ *and* $\bar{b}=\bar{a}$.

To be precise, this is a criterion rather than a lemma. There are certainly valued fields which do not satisfy it (see 1.4.1). Those which do are called *henselian* valued fields. Hensel's Lemma would more accurately refer to the statement that a maximally complete valued field (with respect to the topology) is henselian. These include the p-adics $\mathbb{Q}_p$, and Hahn fields. In particular, the Hahn field example shows that a henselian valued field can be constructed with any field as residue field and any ordered abelian group as value group.

Example 1.4.1. Consider the polynomial $P(x) = x^2 - 13$. P has no solution in $\mathbb{Q}$, of course, although in the 3-adic valuation $\operatorname{res}(P(1)) = 0$ and $\operatorname{res}(P'(1)) = \operatorname{res}(2) \neq 0$ in R.

P does, however, have a root in $\mathbb{Q}_3$. To find it, in $\varprojlim \mathbb{Z}/3^n\mathbb{Z}$ write

$$x = \langle x_1, x_2, x_3, \ldots \rangle$$

with $x_n \in \mathbb{Z}/3^n\mathbb{Z}$ and $x_{n+1} \equiv x_n \pmod{3^n}$ for all n. It is required, for each n, that $x_n^2 - 13 \equiv 0 \pmod{3^n}$. This can be solved recursively by setting $x_1 = 1$ and

$$x_{n+1} \equiv x_n - \frac{P(x_n)}{P'(x_n)} \equiv x_n - \frac{x_n^2 - 13}{2x_n} \pmod{3^{n+1}}$$

to get $x = \langle 1, 7, 16, 16, 259, \ldots \rangle$.

Indeed, by induction it is shown that each x_n is prime to 3 (and therefore a unit modulo 3^{n+1}), whence

$$x_{n+1}^2 - 13 \equiv \left(x_n - \frac{x_n^2 - 13}{2x_n} \right)^2 - 13 \equiv \left(\frac{x_n^2 - 13}{2x_n} \right)^2 \equiv 0 \pmod{3^{n+1}},$$

the last congruence following from $3^n \mid x_n^2 - 13$ and $\gcd(2x_n, 3) = 1$.

Example 1.4.1 suggests a proof of Hensel's Lemma for $\mathbb{Q}_p$, or indeed for any maximally complete valued field, completeness being required to guarantee the existence of a limit of the sequence of approximate roots x_n. It can be shown by a simple cardinality argument, however, that not all henselian valued fields are complete.

Hensel's Lemma lends itself well to reformulation, and there are many equivalent statements in valued fields. See Ribenboim [26] for a thorough exposition of these. Among them is:

Proposition 1.4.2. *Let* (K, v) *be henselian and* $P(x) \in \mathcal{O}[x]$*,* $a \in \mathcal{O}$ *such that* $v(P(a)) > 2v(P'(a))$*. Then there exists* $b \in \mathcal{O}$ *with* $P(b) = 0$ *and* $\bar{b} = \bar{a}$*.* □

To the list we add a couple more equivalent forms of Hensel's Lemma. The first strengthens the condition that $\bar{b} = \bar{a}$.

Proposition 1.4.3. *Let* (K, v) *be henselian and* $P(x) \in \mathcal{O}[x]$*,* $a \in \mathcal{O}$ *such that* $v(P(a)) > 0$ *and* $v(P'(a)) = 0$*. Then there is* $b \in \mathcal{O}$ *with* $P(b) = 0$ *and* $v(a - b) = v(P(a))$*.*

Proof. Taking b as in Hensel's Lemma, If we factor $P(x) = (x - b)Q(x)$, then $v(P(a)) = v(a - b) + v(Q(a))$, whereas

$$0 = v(P'(a)) = v(Q(a) + (a - b)Q'(a)) \geq \min\{v(Q(a)), v(a - b) + v(Q'(a))\} \geq 0.$$

Since $v(a - b) > 0$, we must have $v(Q(a)) = 0$. This gives $v(a - b) = v(P(a))$ as required. □

Now 1.4.3 is used to produce a statement which works in residue characteristic 0 for polynomials over K, and produces roots with the same leading term as the approximate root.

Proposition 1.4.4. *Let* (K, v) *be henselian with* $\operatorname{char}(R) = 0$*. Suppose* $f(x) = \sum\limits_{i=0}^{d} a_i(x - \alpha)^i \in K[x]$ *and* $\xi \in K$ *are such that*

$$\begin{aligned} v(f(\xi)) - \min_{0 \leq i \leq d}\{v(a_i(\xi - \alpha)^i)\} &> \delta \geq 0, \\ v(f'(\xi)) - \min_{1 \leq i \leq d}\{v(ia_i(\xi - \alpha)^{i-1}\} &= 0. \end{aligned} \tag{1.1}$$

Then there exists $\beta \in K$ *such that* $f(\beta) = 0$ *and* $\operatorname{rv}_\delta(\beta - \alpha) = \operatorname{rv}_\delta(\xi - \alpha)$*.*

Proof. Choose m such that $v(a_m(\xi - \alpha)^m) = \min_{i \leq d}\{v(a_i(\xi - \alpha)^i)\}$. We may assume that $m \neq 0$, since if $v(a_0) < v(a_i(\xi - \alpha)^i)$ for all i, $f(\xi) = v(a_0)$ contradicts the inequality in (1.1).

Now set $\sigma := a_m(\xi - \alpha)^m$ and define a new polynomial

$$P(x) := \frac{1}{\sigma} f((\xi - \alpha)x + \alpha) = \frac{1}{\sigma} \sum_{i=0}^{d} a_i(\xi - \alpha)^i x^i.$$

It is easily verified that $P(x) \in \mathcal{O}[x]$ (by choice of m), $v(P(1)) = v(f(\xi)) - v(\sigma) > \delta$, and

$$v(P'(1)) = v\left((\xi-\alpha)\sum_{i=1}^{d} ia_i(\xi-\alpha)^{i-1}\right) - v(\sigma) = v(f'(\xi)) - v(a_m(\xi-\alpha)^{m-1}). \quad (1.2)$$

We claim that in fact $v(P'(1)) = 0$. Since $v(a_m(\xi-\alpha)^m) \leq v(a_i(\xi-\alpha)^i)$ for all $i \leq d$, we have also $v(a_m(\xi-\alpha)^{m-1}) = v(ma_m(\xi-\alpha)^{m-1})$ and

$$v(a_m(\xi-\alpha)^{m-1}) \leq v(a_i(\xi-\alpha)^{i-1}) = v(ia_i(\xi-\alpha)^{i-1})$$

for all $i \leq d$, using here $v(n) = 0$ for all $n \in \mathbb{N}$ since $\operatorname{char}(R) = 0$. This shows that $v(a_m(\xi-\alpha)^{m-1}) = \min_{1\leq i\leq d}\{v(ia_i(\xi-\alpha)^{i-1}\}$. Now (1.1) and (1.2) give $v(P'(1)) = 0$.

Applying Hensel's Lemma (in the form of Proposition 1.4.3) to P at 1, there exists $u \in \mathcal{O}$ such that $P(u) = 0$ and $v(u-1) = v(P(1)) > \delta$. Let $\beta = (\xi-\alpha)u + \alpha$. It follows that $f(\beta) = \sigma P(u) = 0$. Furthermore,

$$v(\beta-\xi) = v((\xi-\alpha)u - (\xi-\alpha)) = v((\xi-\alpha)(u-1)) > v(\xi-\alpha) + \delta.$$

It follows that $v((\beta-\alpha)-(\xi-\alpha)) > v(\xi-\alpha)+\delta$, giving $\mathrm{rv}_\delta(\beta-\alpha) = \mathrm{rv}_\delta(\xi-\alpha)$ by Proposition 1.3.4. □

Chapter 2

Swiss Cheese Decomposition

The form of Hensel's Lemma given in Proposition 1.4.4 suggests a pivotal role, in evaluating a polynomial $f(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i$ near ξ, played by the difference $v(f(\xi)) - \min\{v(a_i(\xi-\alpha)^i)\}$. Indeed, when $v(f(\xi)) = \min\{v(a_i(\xi-\alpha)^i)\}$, Proposition 1.3.6 shows that

$$\mathrm{rv}_\delta(f(\xi)) = \sum_{i=0}^{d} \mathrm{rv}_\delta(a_i)\,\mathrm{rv}_\delta(\xi-\alpha)^i.$$

In the first section, we study more closely what happens when the value of $f(x)$ is strictly greater than the minimum value of its terms $v(a_i(x-\alpha)^i)$. The second section employs this to give an effective means of evaluating $\mathrm{rv}_\delta(f(x))$.

We work over a henselian valued field (K, v) with $\mathrm{char}(K) = 0$.

2.1 Collisions

Definition 2.1.1. For $f(x) \in K[x]$, $f(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i$ has a *collision at* β *around* α if $v(f(\beta)) > \min_{i \leq d}\{v\left(a_i(\beta-\alpha)^i\right)\}$. The *severity* of the collision is the difference $v(f(\beta)) - \min_{i \leq d}\{v\left(a_i(\beta-\alpha)^i\right)\}$.

Note that this concept of collision is not intrinsic to the polynomial $f(x)$ alone, but depends also on α. Thus if $f(x)$ has a collision at β around α, it may be that if

$f(x)$ is recentered to

$$f(x) = \sum_{i=0}^{d} z_i (x-\zeta)^i$$

a collision at β may be avoided.

On the other hand, for any x where $f(x)$ does not have a collision,

$$\mathrm{rv}_\delta(P(x)) = \sum_{i=0}^{d} \mathrm{rv}_\delta(a_i)\, \mathrm{rv}_\delta(x-\alpha)^i$$

is well-defined (by 1.3.6). Thus, having a collision at β around α depends only on $\mathrm{rv}(\beta-\alpha)$. The next proposition shows that there are finitely many leading terms where a collision can happen.

Proposition 2.1.2. *Let* $\alpha \in K$ *and* $f(x) = \sum\limits_{i=0}^{d} a_i(x-\alpha)^i$. *There are finitely many leading terms* $\mathrm{rv}(\beta-\alpha)$ *for which* f *has a collision at* β *around* α.

Proof. There are at most $d(d+1)/2$ values $\delta \in V$ for which f has a collision at β with $v(\beta-\alpha) = \delta$.[1] This is because, recalling Proposition 1.1.4, there can only be a collision if $v(a_i(\beta-\alpha)^i) = v(a_j(\beta-\alpha)^j)$ for some $0 \le i < j \le d$, and this equation has at most one solution for $v(\beta-\alpha)$ in V.

So, fixing a $\delta \in V$, we show that there are finitely many leading terms $\mathbf{u}$ such that $v(\mathbf{u}) = \delta$ and there is a collision at some β with $\mathrm{rv}(\beta-\alpha) = \mathbf{u}$. Pick any β with $v(\beta-\alpha) = \delta$, let $k \le d$ be such that $v(a_k(\beta-\alpha)^k) = v(a_k) + k\delta = \min\limits_{i \le d}\{v(a_i) + i\delta\}$, and define

$$P(x) := \frac{f((\beta-\alpha)x+\alpha)}{a_k(\beta-\alpha)^k} = \frac{1}{a_k(\beta-\alpha)^k}\sum_{i=0}^{d} a_i(\beta-\alpha)^i x^i.$$

$P(x) \in \mathcal{O}[x]$, and the residue polynomial $\overline{P}$ is nonzero. If f has a collision at $\tilde{\beta}$ around α, and $v(\tilde{\beta}-\alpha) = \delta$, then setting $u := \frac{\tilde{\beta}-\alpha}{\beta-\alpha}$ we have $v(u) = 0$ and

$$v\left(P(u)\right) = v\left(\sum_{i=0}^{d} a_i(\tilde{\beta}-\alpha)^i\right) - \min_{i \le d}\left\{v\left(a_i(\tilde{\beta}-\alpha)^i\right)\right\} > 0.$$

So $\bar{u}$ is a root of $\overline{P}$ in R.

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[1] In fact, there are at most d, though it takes somewhat more effort to show this.

Furthermore, if $\mathrm{rv}(\tilde{\beta}-\alpha) \neq \mathrm{rv}(\beta'-\alpha)$ and $w := \frac{\beta'-\alpha}{\beta-\alpha}$, then $v(\tilde{\beta}-\beta') = \delta$ gives

$$v(u-w) = v\left(\frac{\tilde{\beta}-\beta'}{\beta-\alpha}\right) = 0.$$

So $\bar{u} \neq \bar{w}$, and $\tilde{\beta}$ and β' give rise to different roots of $\overline{P}$. Since there are only finitely many of these, the conclusion follows. □

In residue characteristic 0, we can go further by locating collisions near roots of the derivatives of $f(x)$. Here let us introduce the convention that if $\deg(f) = d$ then by *derivatives of* f we mean $f(x)$, $f'(x)$, …, and $f^{(d)}(x)$, notably including f itself as the 0^{th} derivative.

Proposition 2.1.3. *Assume* $\mathrm{char}(R) = 0$. *Suppose* $f(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i$ *has a collision at* β *around* α. *Then there are* $n < d$ *and* $\lambda \in K$ *with*

(i) $f^{(n)}(\lambda) = 0$, *and*

(ii) $\mathrm{rv}(\lambda-\alpha) = \mathrm{rv}(\beta-\alpha)$, *and in particular,* $v(\lambda-\beta) > v(\beta-\alpha)$.

Proof. First note that $\beta \neq \alpha$, as otherwise the inequality in Definition 2.1.1 could not be satisfied. Let m be maximal such that $\min_{i \leq d}\{v(a_i(\beta-\alpha)^i)\} = v(a_m(\beta-\alpha)^m)$, and define $\sigma := a_m(\beta-\alpha)^m$ and $P(x) := \frac{1}{\sigma}f((\beta-\alpha)x+\alpha)$. So, as in 1.4.4, $P \in \mathcal{O}[x]$, and $v(P(1)) > 0$.

Consider $P^{(m)}(1)$. Since

$$P^{(m)}(x) = \frac{1}{\sigma}\sum_{i=m}^{d} \frac{i!}{(i-m)!}a_i(\beta-\alpha)^i x^{i-m},$$

for $i = m$ we have

$$v\left(\frac{1}{\sigma}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i 1^{i-m}\right) = v\left(\frac{m!}{\sigma}a_m(\beta-\alpha)^m\right) = v(m!) = 0 \tag{2.1}$$

while for $i > m$

$$v\left(\frac{1}{\sigma}\frac{i!}{(i-m)!}a_i(\beta-\alpha)^i 1^{i-m}\right) = v\left(a_i(\beta-\alpha)^i\right) - v\left(a_m(\beta-\alpha)^m\right) > 0 \tag{2.2}$$

by the maximality of m. (Note that here we are again using $v(n) = 0$ for all $n \in \mathbb{Z}$, a consequence of $\mathrm{char}(R) = 0$.) Thus $v(P^{(m)}(1)) = 0$.

Now let n be least such that $v(P^{(n+1)}(1)) = 0$. The above shows that n is at most $m - 1$. Applying Hensel's Lemma to $P^{(n)}$, there is a $u \in \mathcal{O}$ with $\bar{u} = \bar{1} \in R$ and $P^{(n)}(u) = 0$. It follows that $v(u - 1) > 0 = v(1)$, so that $\mathrm{rv}(u) = \mathrm{rv}(1)$.

Let $\lambda := u(\beta - \alpha) + \alpha$. So $\mathrm{rv}(\lambda - \alpha) = \mathrm{rv}(u)\,\mathrm{rv}(\beta - \alpha) = \mathrm{rv}(\beta - \alpha)$. This implies that $v(\lambda - \beta) = v((\lambda - \alpha) - (\beta - \alpha)) > v(\beta - \alpha)$ as in (ii). Finally,

$$0 = P^{(n)}(u) = \frac{(\beta - \alpha)^n}{\sigma} f^{(n)}((\beta - \alpha)u + \alpha) = \frac{(\beta - \alpha)^n}{\sigma} f^{(n)}(\lambda).$$

Since $\alpha \neq \beta$, $f^{(n)}(\lambda) = 0$. □

The situation when $\mathrm{char}(R) = p$ is more complicated. Comparing the calculations in (2.1) and (2.2) above, we find $v(P^{(m)}(1)) = v(m!)$ (note that $v(i!/(i-m)!) \geq v(m!)$, since $m!$ divides $i!/(i-m)!$). If $v(P(1)) > 2^m v(m!)$, Hensel's Lemma (in the form of Proposition 1.4.2) would still apply to $P^{(n)}$ for whichever n has $v(P^{(n)}(1)) > 2v(P^{(n+1)}(1))$, and so we could find a root λ as before.

Otherwise, if $v(P(1)) \leq 2^m v(m!)$, the same argument will work for any $\tilde{\beta}$ such that $\mathrm{rv}(\tilde{\beta} - \alpha) = \mathrm{rv}(\beta - \alpha)$ and

$$\widetilde{P}(x) = \frac{1}{a_m(\tilde{\beta} - \alpha)^m} f((\tilde{\beta} - \alpha)x + \alpha).$$

So if there is no root λ of a derivative of $f(x)$, it must be that $\widetilde{P}(1) \leq 2^m v(m!)$ for every $\tilde{\beta} \in B_{>v(\beta - \alpha)}(\beta)$. Therefore

$$v(f(\tilde{\beta})) = v(a_m(\tilde{\beta} - \alpha)^m) + v(\widetilde{P}(1)) \leq \min_{i \leq d} \left\{ v(a_i(\tilde{\beta} - \alpha)^i) \right\} + v\left((m!)^{2^m}\right)$$

and we have proven that the collision has severity bounded by $v\left((d!)^{2^d}\right)$:

Proposition 2.1.4. *Assume* $\mathrm{char}(R) = p > 0$. *If* $f(x) = \sum_{i=0}^{d} a_i(x - \alpha)^i$ *has a collision at* β *around* α, *then either*

(i) there exists a root λ *of a derivative of* f *as in Proposition 2.1.3, or*

(ii) there is an integer $q > 0$ *such that* $\mathrm{rv}(x-\alpha) = \mathrm{rv}(\beta - \alpha)$ *implies*

$$\min_{i\leq d}\left\{v\left(a_i(x-\alpha)^i\right)\right\} < v(f(x)) \leq \min_{i\leq d}\left\{v\left(a_i(x-\alpha)^i\right)\right\} + v(q).$$

Moreover, q *can be chosen no larger than* $(d!)^{2^d}$. □

2.2 The decomposition

Propositions 2.1.3 and 2.1.4 give a handle on what is happening near a collision. We exploit this to prove the existence of a decomposition of K relative to a polynomial $f(x) \in K[x]$, on each piece of which $f(x)$ can be rewritten in such a way as to avoid collisions. The decomposition is given in blocks of swiss cheese.

Definition 2.2.1. A *swiss cheese* is a set of the form $A \setminus (B_1 \cup \ldots \cup B_n)$, where $A, B_1, \ldots, B_n$ are (open or closed) balls with $B_i \subseteq A$.

Like the m in the proof of Proposition 2.1.3, we will frequently need to refer to the largest degree term carrying the smallest valuation. Therefore define

$$m(f,\alpha,S) := \max\left\{i \leq d \mid \exists x \in S\ \forall j \leq d\ \left(v(a_i(x-\alpha)^i) \leq v(a_j(x-\alpha)^j)\right)\right\} \quad (2.3)$$

where the a_i are the coefficients of the expansion of f around α,

$$f(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i.$$

Thus, $m(f,\alpha,S)$ is the highest order term in f centered at α which can have minimal valuation (among the other terms of f) on S.

As before, we first prove the main result for residue characteristic 0, and then indicate the modifications needed if $\mathrm{char}(R) = p$.

Proposition 2.2.2. *Suppose* $\mathrm{char}(R) = 0$. *Let* $f(x) \in K[x]$ *and* S *be a swiss cheese in* K. *Then there exist (disjoint) sub-swiss cheeses* $T_1, \ldots, T_k \subseteq S$ *and* $\alpha_1, \ldots, \alpha_k \in K$ *such that*

$$S = \dot{\bigcup} T_i$$

and for all $x \in T_i$,

$$v(f(x)) = v(a_{im_i}(x-\alpha_i)^{m_i}),$$

with $f(x) = \sum_{n=0}^{d} a_{in}(x-\alpha_i)^n$ *and* $m_i = m(f, \alpha_i, T_i)$.

Furthermore, $\alpha_1, \dots, \alpha_k$ *can be chosen algebraic over the subfield of* K *generated by the coefficients of* $f(x)$.

Proof. For simplicity, assume S is a ball $B_{\geq\gamma}(\alpha)$ (or K). No generality is lost as a decomposition for $B_{\geq\gamma}(\alpha) \supseteq S$ may simply be intersected with S to get the desired result.

Let $f(x) = \sum_{n=0}^{d} a_i(x-\alpha)^i$. The proof proceeds by induction on $m(f, \alpha, S)$. Clearly, if $m(f, \alpha, S) = 0$, then by 1.1.4 $v(f(x)) = v(a_0)$ for all $x \in S$.

Now suppose $m(f, \alpha, S) = m$. Let

$$D := \{\delta \geq \gamma \mid \forall i \leq m\, (v(a_m) + m\delta \leq v(a_i) + i\delta)\}.$$

In other words, $m(f, \alpha, S) = m$ when $v(a_m(x-\alpha)^m)$ is minimal somewhere in S, while D gives those values where it actually is minimal.

D is an initial segment of $[\gamma, \infty)$. Indeed, if $\gamma \leq \varepsilon < \delta \in D$ and $i < m$, then $v(a_i) + i\delta \geq v(a_m) + m\delta$ implies

$$v(a_i) + i\varepsilon > v(a_m) + m\varepsilon \tag{2.4}$$

so $\varepsilon \in D$. We need not consider $i > m$, by the maximality of m.

In particular, the inequality in (2.4) becomes strict for $\varepsilon < \delta$. Therefore we have also shown that if $\delta \in D$ is not a maximal element of D, then for all x such that $v(x-\alpha) = \delta$,

$$v(f(x)) = v(a_m(x-\alpha)^m) \tag{2.5}$$

since $v(a_m(x-\alpha)^m) < v(a_i(x-\alpha)^i)$ for all $i \neq m$.

Suppose first that D has no maximal element, and define

$$B_D := \{x \in S \mid v(x-\alpha) \in D\}.$$

Then there is some $i < m$ such that $v(a_m(x-\alpha)^m) > v(a_i(x-\alpha)^i)$ whenever $v(x-\alpha) > \delta$ for every $\delta \in D$. Set $\eta := v(a_i) - v(a_m)$ and note that

$$S \setminus B_D = B_{\geq\gamma}(\alpha) \setminus B_D = B_{>\eta/(m-i)}(\alpha)^2$$

since for $x \in S$,

$$\begin{aligned} x \notin B_D &\Leftrightarrow v(a_i(x-\alpha)^i) < v(a_m(x-\alpha)^m) \\ &\Leftrightarrow v(a_i) - v(a_m) < (m-i)v(x-\alpha). \end{aligned}$$

Therefore, if $x \in B_D$ then $v(f(x)) = v(a_m(x-\alpha)^m)$ by (2.5), whereas if $x \in S \setminus B_D$ then $m(P, \alpha, B_{>\eta/(m-i)}(\alpha)) < m$ and the induction hypothesis applies.

On the other hand, suppose $D = [\gamma, \delta]$ has a maximum at δ. Now S decomposes into three swiss cheeses:

$$S \;=\; B_{>\delta}(\alpha) \;\cup\; B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) \;\cup\; B_{\geq\gamma}(\alpha) \setminus B_{\geq\delta}(\alpha).$$

On the last of these, as observed above, $v(f(x)) = v(a_m(x-\alpha)^m)$. On the first, $m(f, \alpha, B_{>\delta}(\alpha)) < m$, so that again, the induction hypothesis applies. It therefore remains only to consider $B_{\geq\delta}(\alpha) \setminus B_{>\delta}(\alpha) =: A$, i.e. when $v(x-\alpha) = \delta$.

Let $C := \{x \in A \mid v(f(x)) \neq v(a_m(x-\alpha)^m)\}$, so C is the set of elements of A at which f has a collision around α. Now, C is the disjoint union of equivalence classes under the equivalence $x \sim y \Leftrightarrow v(x-y) > \delta \Leftrightarrow \mathrm{rv}(x-\alpha) = \mathrm{rv}(y-\alpha)$.

Proposition 2.1.2 shows that there are finitely many such equivalence classes. Furthermore, by Proposition 2.1.3, each of the equivalence classes contains a root λ of a derivative of f.

So, C is a finite union of balls of the form $B_{>\delta}(\lambda)$, having centers algebraic over the coefficients of $f(x)$. Thus we see that $A \setminus C$ is a swiss cheese on which $v(f(x)) = v(a_m(x-\alpha)^m)$, so now it remains only to determine $v(f(x))$ on C.

Choose a $\lambda \in C$, $f^{(n)}(\lambda) = 0$, and let

$$f(x) = \sum_{i=0}^{d} a_i(x-\alpha)^i = \sum_{i=0}^{d} b_i(x-\lambda)^i.$$

---

[2] It should be pointed out here that the failure of D to have a maximal element means that η is not divisible by $m-i$ in V, but as noted earlier we may still define the ball $B_{>\eta/(m-i)}(\alpha)$.

Taking $\sigma := a_m(\lambda-\alpha)^m$, define once more

$$P_\alpha(x) := \frac{f((\lambda-\alpha)x+\alpha)}{\sigma} = \frac{1}{\sigma}\sum_{i=0}^{d} a_i(\lambda-\alpha)^i x^i$$

$$P_\lambda(x) := \frac{f((\lambda-\alpha)x+\lambda)}{\sigma} = \frac{1}{\sigma}\sum_{i=0}^{d} b_i(\lambda-\alpha)^i x^i$$

Note that $P_\alpha(x+1) = P_\lambda(x)$ and that $P_\alpha(x), P_\lambda(x) \in \mathcal{O}[x]$. However, since

$$v(a_i(\lambda-\alpha)^i) > v(a_m(\lambda-\alpha)^m)$$

for $m < i \leq d$, the residue polynomial $\overline{P_\alpha}(x)$ has degree m. Now it follows from $P_\alpha(x+1) = P_\lambda(x)$ that $\overline{P_\lambda}(x)$ also has degree m.

By considering the residues of the coefficients of P_λ, then, we get $v(b_i(\lambda-\alpha)^i) > v(\sigma)$ for $i > m$ and $v(b_m(\lambda-\alpha)^m) = v(\sigma)$. Since $v(\lambda-\alpha) = \delta$, in shifting from $B_{\geq\gamma}(\alpha)$ to $B_{>\delta}(\lambda)$, we see that

$$m(f, \alpha, B_{\geq\gamma}(\alpha)) \geq m(f, \lambda, B_{>\delta}(\lambda)).$$

But equality may occur, so that we cannot yet invoke the induction. Instead, if indeed $m(f, \lambda, B_{>\delta}(\lambda)) = m$, the proof concludes with a second induction on the number k of roots of the nonconstant derivatives of f contained in the ball $B_{>\delta}(\lambda)$.

As above, we decompose $B_{>\delta}(\lambda)$ into the three Swiss cheeses $B_{>\varepsilon}(\lambda)$, $B_{\geq\varepsilon}(\lambda) \setminus B_{>\varepsilon}(\lambda)$, and $B_{\geq\delta}(\lambda) \setminus B_{\geq\varepsilon}(\lambda)$; and as above, each case is dealt with easily except where collisions occur within $B_{\geq\varepsilon}(\lambda) \setminus B_{>\varepsilon}(\lambda) =: A'$.

For the base case $k = 1$ of the induction (not $k = 0$, as $B_{>\delta}(\lambda)$ contains at least the root λ), A' is empty, so we are done. Otherwise, there are strictly fewer than k roots of derivatives of f in A', again because $\lambda \notin A'$, so here the induction step gives the rest of the decomposition.

To finish, note again that the α_i in the statement of the Proposition, as well as the centers of all of the balls appearing in a swiss cheese of the decomposition, were selected from among the roots of the derivatives of $f(x)$. Thus these elements are algebraic over the subfield of K generated by $\{a_0, \ldots, a_d\}$. $\square$

Now, in residue characteristic p, the proof above remains valid except that when looking in C, not every ball necessarily contains a root λ. In those that do not, however, as in Proposition 2.1.4 we can bound by $v(q)$ the severity of a collision within that ball. So the corresponding modification reads

Proposition 2.2.3. *Suppose* $\mathrm{char}(R) = p$. *Let* $f(x) \in K[x]$ *and* S *be a swiss cheese in* K. *Then there exist (disjoint) sub-swiss cheeses* $T_1, \ldots, T_k \subseteq S$ *and* $\alpha_1, \ldots, \alpha_k \in K$ *such that*

$$S = \dot{\bigcup} T_i$$

and for each T_i, *either*

(i)

$$\forall x \in T_i \left(v(f(x)) = v\left(a_{im_i}(x - \alpha_i)^{m_i}\right)\right)$$

(with $f(x) = \sum\limits_{n=0}^{d} a_{in}(x - \alpha_i)^n$ *and* $m_i = m(f, \alpha_i, T_i)$*);*

(ii) or there is an integer $q \in \mathbb{N}$, $0 < q \leq (d!)^{2^d}$, *such that*

$$\forall x \in T_i \left(\min_{j \leq d} \left\{ v\left(a_{ij}(x - \alpha_i)^j\right)\right\} < v(f(x)) \leq \min_{j \leq d} \left\{ v\left(a_{ij}(x - \alpha_i)^j\right)\right\} + v(q) \right).$$

Furthermore, $\alpha_1, \ldots, \alpha_k$ *can be chosen from the relative algebraic closure in* K *of the subfield generated by the coefficients of* $f(x)$. ☐

Finally, we return to the leading term structures to find expressions for analyzing $\mathrm{rv}_\delta(f(x))$. Thanks to Propositions 1.3.6 and 1.3.7, this is an immediate consequence of the above two propositions.

Proposition 2.2.4. *Let* $f(x) = \sum\limits_{i=0}^{d} a_i x^i \in K[x]$ *and* $0 \leq \delta \in V$. *Then there are disjoint swiss cheeses* $U_1, \ldots, U_k$ *partitioning* $K = \bigcup\limits_{i=1}^{k} U_i$ *and* $\alpha_1, \ldots, \alpha_k \in K$ *such that for each* i, *if* $f(x) = \sum\limits_{j=0}^{d} a_{ij}(x - \alpha_i)^j$ *then for all* $x \in U_i$ *either*

(i) $\mathrm{rv}_\delta\left(f(x)\right) = \sum\limits_{j=0}^{d} \mathrm{rv}_\delta(a_{ij})\, \mathrm{rv}_\delta(x - \alpha_i)^j$ *is well-defined, or*

(ii) there is a positive integer $q \leq (d!)^{2^d}$ *such that*

$$\operatorname{rv}_\delta(f(x)) = \operatorname{rv}_\delta\left(\sum_{j=0}^{d} \operatorname{rv}_{\delta+v(q)}(a_{ij}) \operatorname{rv}_{\delta+v(q)}(x-\alpha_i)^j\right)$$

is well-defined.

The latter case only occurs in positive residue characteristic.

Furthermore, the $\alpha_1, \ldots, \alpha_k$ *can be chosen to be algebraic over* $\{a_0, \ldots, a_d\}$.

Proof. Let $K = \bigcup_{i=1}^{k} U_i$ and $\alpha_1, \ldots, \alpha_k \in K$ be the swiss cheese decomposition given by the Propositions. Each U_i is of one of two types as in the statement of 2.2.3.

In case U_i gives an α_i whereby $v(f(x)) = \min_{j \leq d}\{v(a_{ij}(x-\alpha_i)^j)\}$ for all $x \in U_i$, we have

$$\operatorname{rv}_\delta(f(x)) = \sum_{j=0}^{d} \operatorname{rv}_\delta(a_{ij}) \operatorname{rv}_\delta(x-\alpha_i)^j$$

by Proposition 1.3.6

In case U_i has an α_i for which

$$\min_{j \leq d}\left\{v\left(a_{ij}(x-\alpha_i)^j\right)\right\} < v(f(x)) \leq \min_{j \leq d}\left\{v\left(a_{ij}(x-\alpha_i)^j\right)\right\} + v(q)$$

then Proposition 1.3.7 shows that although $\sum \operatorname{rv}_\delta(a_{ij}) \operatorname{rv}_\delta(x-\alpha_i)^j$ is not well-defined,

$$\operatorname{rv}_{\delta+v(q)\to\delta}\left(\sum_{i=0}^{d} \operatorname{rv}_\delta(a_{ij}) \operatorname{rv}_\delta(x-\alpha_i)^j\right)$$

is well-defined. □

Though each of the preceding propositions is stated for a single polynomial $f(x)$, the same results will hold for any finite number of polynomials $f_1(x), \ldots, f_n(x)$. To obtain the desired decomposition, simply apply the proposition to each $f_i(x)$ separately, and then intersect the resulting partitions to get one which works for all $f_i(x)$ simultaneously, using the fact that the intersection of two swiss cheeses is again a swiss cheese.

The methods used in the decomposition given above are reminiscent of those employed by Cohen [7] in his decision procedure for the p-adics. In fact, it will be shown in Chapter 5 that at least in residue characteristic 0, these results and techniques can be used to give an effective quantifier elimination, and therefore a decision procedure, for the field relative to the leading term structures.

One may also compare the cell decomposition for characteristic 0 henselian fields worked out by Cluckers and Loeser in the context of b-minimality. See [6].

Chapter 3

Interpretations

While the first two chapters have been of a purely algebraic character, in this and the following the focus turns to the model theory of henselian valued fields of characteristic 0 and the leading term structures. To begin, in the first section some basic relevant definitions are introduced. Then, various languages used to study valued fields are surveyed and compared to the language of the leading term structures.

3.1 Definitions

A structure interpretable in another can be mimicked by definable sets and operations.

Definition 3.1.1. Let M and N be structures, not necessarily for the same language and possibly multisorted. Let also the language of M consist of sorts $s \in \mathcal{S}$ and symbols $r \in \mathcal{R}$ of arity n_r[1]. (The symbols in $\mathcal{R}$ may be relation, function, or constant symbols, but for convenience we assume they are all relation symbols.) If $A \subseteq N$, then M is *interpretable* in N over A if there are

- A-definable sets $T_s \subseteq N^{k_s}$ for each $s \in \mathcal{S}$;
- A-definable equivalence relations $\sim_s$ on each T_s;

---

[1] More precisely, the arity of a relation symbol in a multisorted language is a function from the set $\{1, 2, \ldots, n_r\}$ to $\mathcal{S}$.

- and surjections $\varepsilon_s : T_s \twoheadrightarrow s^M$ such that $\varepsilon_s(x) = \varepsilon_s(y)$ iff $x \sim_s y$

such that for each $r \in R$, $r^M \subseteq s_1^M \times \ldots \times s_{n_r}^M$, the pullback of r^M via $\varepsilon_{s_1} \times \ldots \times \varepsilon_{s_{n_r}}$ is also A-definable in N.

In other words, the $\sim_s$-equivalence classes act as the s-sorted elements of M, with definable sets on T_s filling in for the relations in M. In the setting of fields, interpretable structures can be thought of as a generalization of algebraic groups.

Example 3.1.2. Let k be a field and $n \geq 1$. The group $\mathrm{GL}_n(k)$ is interpretable in k on the set

$$T = \left\{ \langle a_{ij} \rangle \in k^{n^2} \mid \det(a_{ij}) \neq 0 \right\}$$

with the inverse image of the matrix multiplication being defined in the obvious way.

In this case, the equivalence relation $\sim$ on T is trivial ($\langle a_{ij} \rangle \sim \langle b_{ij} \rangle \Leftrightarrow \forall i, j \leq n, a_{ij} = b_{ij}$). For an example where it would be nontrivial, we could similarly give an interpretation of the projective linear group $\mathrm{PSL}_n(k)$ as a quotient of $\mathrm{SL}_n(k)$.

Example 3.1.3. Consider the valued field $\mathbb{Q}$ under the p-adic valuation, in the language containing

- sorts for the field, the value group, and the residue field;
- the usual ring language on the field and residue field sorts, and the ordered group language on the value group sort;
- and the valuation v and residue map res as maps between sorts (say $\mathrm{res}(x) = 0$ for $x \notin \mathcal{O}$).

This is interpretable in the pure field $\mathbb{Q}$. Recall that the set $\mathbb{Z}$ is definable in $\mathbb{Q}$, as well as the graph of exponentiation $\{\langle x, y, z \rangle \in \mathbb{Z}^2 \mid x^y = z\}$ on $\mathbb{Z}$ (see Robinson [29] for the former fact, Matiyasevich [22] for the latter[2]).

If $\mathbb{Q}$ is interpreted by the identity in itself, and the value group by the definable subset $\mathbb{Z}$, the map $v_p : \mathbb{Q} \to \mathbb{Z}$ is defined by

$$v_p(x) = y \iff \exists a, b \in \mathbb{Z} \left(\gcd(a, p) = \gcd(b, p) = 1 \wedge x = p^y(a/b) \right).$$

---

[2] Definability of exponentiation in $\mathbb{Z}$ is a consequence of Gödel [9], but Matiyasevich achieves it more efficiently.

Now one can interpret the residue field on the set $\mathcal{O} = \{x \in \mathbb{Q} \mid v_p(x) \geq 0\}$ under the equivalence relation $x \sim y \Leftrightarrow v_p(x-y) > 0$. The operations on the residue field are simply the addition and multiplication inherited from $\mathcal{O}$.

As suggested in Chapter 1, the leading term structures in a sense encompass both residue field and value group. This can now be made more explicit. Regarding the language, the leading term structure RV_δ comes with the leading term multiplication and partial addition in the form of the relation $\oplus_\delta$.

Proposition 3.1.4. *Let (K, v) be a valued field with leading term structure* RV.

1. *The value group V is interpretable in* RV.
2. *The residue field R is interpretable in* RV.
3. *For $\gamma > \delta \geq 0$ in V, RV_δ is interpretable in RV_γ as long as we may use as a parameter an element $\mathbf{d} \in \mathrm{RV}_\gamma$ of value $v(\mathbf{d}) = \delta$.*

Proof. (1): To begin, observe that $v(\mathbf{x}) > 0$ is definable in RV. Indeed,

$$v(\mathbf{x}) > 0 \Longleftrightarrow \mathbf{x} + \mathrm{rv}(1) = \mathrm{rv}(1).$$

From this it follows that $v(\mathbf{x}) = 0$ is also definable:

$$v(\mathbf{x}) = 0 \Longleftrightarrow \neg v(\mathbf{x}) > 0 \wedge \exists \mathbf{y}\left(\mathbf{x}\mathbf{y} = \mathrm{rv}(1) \wedge \neg v(\mathbf{y}) > 0\right).$$

Now define the equivalence relation $\sim$ on RV by

$$\mathbf{x} \sim \mathbf{y} \Longleftrightarrow \exists \mathbf{c}\left(v(\mathbf{c}) = 0 \wedge \mathbf{x} = \mathbf{c}\mathbf{y}\right).$$

Clearly, we have $\mathbf{x} \sim \mathbf{y}$ iff $v(\mathbf{x}) = v(\mathbf{y})$, so that the equivalence classes of $\sim$ in RV are in bijection with V.

Moreover, addition of $v(\mathbf{x}) + v(\mathbf{y})$ in V corresponds to the multiplication $\mathbf{x}\mathbf{y}$ in RV, and the group ordering $<$ is defined by $\mathbf{x} < \mathbf{y}$ iff $\mathbf{x} \neq \infty \wedge \mathbf{x} + \mathbf{y} = \mathbf{x}$.

(2): Nonzero elements of the residue field are in bijection with $\mathbf{x} \in \mathrm{RV}$ such that $v(\mathbf{x}) = 0$, with $\bar{x}$ corresponding to $\mathrm{rv}(x)$. To see this, simply note that when $v(x) = v(y) = 0$,

$$\bar{x} = \bar{y} \Longleftrightarrow v(x-y) \in \mathfrak{m} \Longleftrightarrow \mathrm{rv}(x) = \mathrm{rv}(y).$$

The multiplication and addition of R on these elements is the same as the multiplication and addition in RV except the case where two elements of R add to 0. Dealing with this special case, however, is routine. For example one may map $\infty \in \mathrm{RV}$ to $0 \in R$.

(3): Considering $\mathbf{x} = \mathrm{rv}_\gamma(x), \mathbf{y} = \mathrm{rv}_\gamma(y) \in \mathrm{RV}_\gamma$, it will be enough to show that $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y)$ is definable in RV_γ. Recalling 1.3.8, this follows from

$$\begin{aligned} \mathrm{rv}_\delta(x) = \mathrm{rv}_\delta(y) &\iff v(x-y) > v(y) + \delta \\ &\iff \forall \mathbf{z} \in \mathrm{RV}_\gamma \left(\oplus_\gamma(\mathbf{x}, -\mathbf{y}, \mathbf{z}) \to v(\mathbf{z}) > v(\mathbf{y}) + \delta\right). \end{aligned}$$

□

By itself, then, RV can be thought of as a kind of amalgam of residue field and value group. The above shows that R and V are interpretable in RV, and we will show that in the presence of a cross section of the value group in RV, RV is interpretable in (R, V). But in fact, if the cross section is definable, then a stronger equivalence holds.

Definition 3.1.5. A subset $C \subseteq \mathrm{RV}$ is a *cross section* if for every $\delta \in V$ there is precisely one $\mathbf{c}_\delta \in C$ such that $v(\mathbf{c}_\delta) = \delta$, and for all $\delta, \gamma \in V$, $\mathbf{c}_\delta \mathbf{c}_\gamma = \mathbf{c}_{\delta+\gamma}$.

Proposition 3.1.6. *Suppose* RV *has a cross section* $C = \{\mathbf{c}_\delta \mid \delta \in V\}$. *Then* RV *is interpretable in the two sorted structure* (R, V), *the first sort with the ring language and the second with the ordered group language.*

Proof. Let $S = (R^\times \times V) \cup \{\langle 0, 0\rangle\}$, i.e. S is the subset of the cartesian product of R and V where the first coordinate is only 0 if the second is 0 as well. Define $\varepsilon : S \to \mathrm{RV}$ by

$$\varepsilon(\langle \bar{x}, \delta\rangle) = \begin{cases} \mathbf{c}_\delta \, \mathrm{rv}(x) & \text{if } \mathrm{res}(x) = \bar{x} \neq 0 \\ \infty & \text{if } \langle \bar{x}, \delta\rangle = \langle 0, 0\rangle. \end{cases}$$

We claim that ε is a bijection. Indeed, if $\mathbf{x} \in \mathrm{RV}$ is not ∞, then $v(\mathbf{x}) = \delta \neq \infty$, and there is $x \in K$ with $\bar{x} \neq 0$ in R and $\mathbf{x}/\mathbf{c}_\delta = \mathrm{rv}(x)$. Thus $\mathbf{x} = \varepsilon(\langle \bar{x}, \delta\rangle)$ and ε is onto. Conversely, if $\mathbf{c}_\delta \, \mathrm{rv}(x) = \mathbf{c}_\gamma \, \mathrm{rv}(y)$ with $v(x) = v(y) = 0$, then $v(\mathbf{c}_\delta \, \mathrm{rv}(x)) = \delta =$

$v(\mathbf{c}_\gamma \operatorname{rv}(y)) = \gamma$. So $\mathbf{c}_\delta = \mathbf{c}_\gamma$ and $\operatorname{rv}(x) = \operatorname{rv}(y)$. Now $\operatorname{res}(x/y) = 1$ (by 1.3.3) implies $\bar{x} = \bar{y}$, and so ε is injective as well.

It remains to show that the inverse image of multiplication in RV and the $\oplus$ relation are definable in S. But since $(\mathbf{c}_\delta \operatorname{rv}(x))(\mathbf{c}_\gamma \operatorname{rv}(y)) = \mathbf{c}_{\delta+\gamma} \operatorname{rv}(xy)$, the inverse image of the graph of multiplication on nonzero elements is just $\{\langle\langle\bar{x}, \delta\rangle, \langle\bar{y}, \gamma\rangle, \langle\bar{x}\bar{y}, \delta + \gamma\rangle\rangle\}$. If either of the factors is $\langle 0, 0\rangle$, of course, the product must also be $\langle 0, 0\rangle$. So the multiplication relation is clearly definable in S.

For $\oplus$, if $\mathbf{c}_\delta \operatorname{rv}(x) + \mathbf{c}_\gamma \operatorname{rv}(y) = \mathbf{c}_\beta \operatorname{rv}(z)$ (with $v(x) = v(y) = v(z) = 0$) holds in RV, there are several possibilities to consider:

- If say $\delta < \gamma$, then $\delta = \beta$ and $\operatorname{rv}(x) = \operatorname{rv}(z)$.
- Likewise if $\gamma > \delta$.
- If $\delta = \gamma = \beta$, then the addition is well-defined and $\bar{z} = \bar{x} + \bar{y} \neq 0$.
- Finally, if $\beta > \min\{\delta, \gamma\}$, the addition is not well-defined and the addition relation holds for *any* $\mathbf{c}_\beta \operatorname{rv}(z)$ such that $\beta > \min\{\delta, \gamma\}$ (see Proposition 1.3.8).

Since each of these cases is clearly definable in S, the inverse image of $\oplus$ via ε is definable and the interpretation holds. □

Although with the existence of the cross section, RV and (R, V) are mutually interpretable, there is not a complete correspondence between the definable sets in the two structures. If C is not definable in RV, then while there is a definable subset of $R \times V$ isomorphic to V, in RV there is only an equivalence relation whose equivalence classes are isomorphic to V. Mutual interpretability is a weaker relation than *biinterpretability*.

Definition 3.1.7. Suppose M is interpretable in N via the definable $S \subseteq N^k$ and surjection $\varepsilon : S \twoheadrightarrow M$, and that N is interpretable in M via the definable $T \subseteq M^\ell$ and surjection $\vartheta : T \twoheadrightarrow N$. These interpretations form a *biinterpretation* if the composite maps $\varepsilon \circ \vartheta$ and $\vartheta \circ \varepsilon$ (restricted to the domains where the composites make sense) are definable in M and N respectively.

The distinction between two structures being mutually interpretable and biinterpretable is a case of two structures not only being able to imitate each other, but to imitate each other so well that each structure can definably recognize the other's imitation of it.

Proposition 3.1.8. *If there is a definable cross section C in* RV*, then there is a biinterpretation between* RV *and* (R, V)*.*

Proof. Let $R^* := \{\mathbf{x} \in \mathrm{RV} \mid v(\mathbf{x}) = 0\} \cup \{\infty\}$. We have on the one hand the bijection $\varepsilon : S \to \mathrm{RV}$ from 3.1.6, and on the other hand two bijections

$$\vartheta_R : R^* \to R$$

$$\vartheta_V : C \to V$$

which interpret the sorts R and V in RV as in 3.1.4 (the map ϑ_V is different from that in 3.1.4 in the absence of C, but the argument is the same).

Now, the composition $\varepsilon \circ (\vartheta_R \times \vartheta_V) : (R^* \times C) \to \mathrm{RV}$ sends $\langle \mathbf{x}, \mathbf{c}_\delta \rangle$ to $\mathbf{x}\mathbf{c}_\delta$, which is obviously definable in RV. The map $\vartheta_R \circ \varepsilon$ is defined on the subset of S for which the second coordinate is 0, and sends $\langle \bar{x}, 0 \rangle$ to $\bar{x}$. The map $\vartheta_V \circ \varepsilon$ is defined on the subset of S for which the first coordinate is 1 and sends $\langle 1, \delta \rangle$ to δ. These are trivially definable in $S \times R$ and $S \times V$ respectively. Thus we have a biinterpretation. □

3.2 On Languages

A number of different languages have been proposed for working with valued fields in first-order logic. As a single sorted structure, one adds to the ring language $\mathcal{L}_R = \{0, 1, +, \times\}$ a predicate for the valuation ring $\mathcal{O}$. This is in fact enough to specify the valuation, since $v(x) \leq v(y)$ iff $y/x \in \mathcal{O}$. It is then straightforward to show that the value group and residue field are interpretable in the field in this language. Other authors have used the 3-sorted language as in Example 3.1.3.

On the other hand, one may also build a language based around the leading term structures. The *leading term language* refers to a multisorted language

$$(K, \langle \mathrm{RV}_\delta \rangle_{\delta \in \Delta})$$

with the ring language on the field sort, $\Delta \subseteq \{\delta \in V \mid 0 \leq \delta < \infty\}$ to be specified as needed, the multiplication and the addition relation $\oplus_\delta$ on each RV_δ, and as maps between the sorts $\mathrm{rv}_\delta : K \to \mathrm{RV}_\delta$ and $\mathrm{rv}_{\gamma\to\delta} : \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$ for each $\gamma \geq \delta \in \Delta$.

This is a simplification of the language of additive and multiplicative congruences (*amc-structures*) used by Kuhlmann [20]. He defines, for each $\delta \geq 0$ in V, the system

$$K_\delta := (\mathcal{O}_\delta, G_\delta, \Theta_\delta(x, y))$$

whereby $\mathcal{O}_\delta := \mathcal{O}/\mathfrak{m}_\delta$, $G_\delta := K^\times/(1 + \mathfrak{m}_\delta)$ ($= \mathrm{RV}_\delta$ as a multiplicative group). The notations π_δ and π^*_δ are used for the canonical maps $\mathcal{O} \to \mathcal{O}_\delta$ and $K^\times \to G_\delta$ respectively (again, π^*_δ is the same as our rv_δ). Then the relation Θ_δ is defined on $\mathcal{O}_\delta \times G_\delta$ as

$$\Theta_\delta(x, y) \Longleftrightarrow \exists z \in \mathcal{O} \ \ (\pi_\delta(z) = x \wedge \pi^*_\delta(z) = y)\,.$$

In fact, we show that the leading term structures and Kuhlmann's amc-structures are biinterpretable.

Proposition 3.2.1. *For $\delta \geq 0$, K_δ is interpretable in* RV_δ.

Proof. Since G_δ is a reduct of RV_δ, it only remains to give an interpretation of $\mathcal{O}_\delta$, as well as of the relation Θ_δ.

Elements $a, b \in \mathcal{O}$ have equal images in $\mathcal{O}_\delta$ if (and only if) $a - b \in \mathfrak{m}_\delta$, that is, if $v(a - b) > \delta$. Using the notation

$$\mathrm{RV}^+_\delta := \mathrm{RV}_\delta(\mathcal{O}) = \{\mathbf{x} \in \mathrm{RV}_\delta \mid 0 \leq v(\mathbf{x})\}$$

define an equivalence relation $\sim$ on RV^+_δ by

$$\mathbf{a} \sim \mathbf{b} \Longleftrightarrow \forall \mathbf{c} \in \mathrm{RV}_\delta\, (\mathbf{a} - \mathbf{b} = \mathbf{c} \to v(\mathbf{c}) > \delta)\,.$$

It is clear that there is a bijection between $\mathcal{O}_\delta$ and the equivalence classes of $\sim$ on RV^+_δ.

In fact, if we define the surjection

$$\vartheta : \mathrm{RV}^+_\delta \twoheadrightarrow \mathcal{O}_\delta$$

so that $\vartheta(\mathbf{x})$ is the element of $\mathcal{O}_\delta$ corresponding to the equivalence class of $\mathbf{x}$, then we have $\pi_\delta = \vartheta \circ \mathrm{rv}_\delta$.

It remains to show that the graphs of addition

$$A := \left\{ \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\mathrm{RV}_\delta^+)^3 \mid \vartheta(\mathbf{x}) + \vartheta(\mathbf{y}) = \vartheta(\mathbf{z}) \right\}$$

and multiplication

$$M := \left\{ \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\mathrm{RV}_\delta^+)^3 \mid \vartheta(\mathbf{x})\vartheta(\mathbf{y}) = \vartheta(\mathbf{z}) \right\}$$

as well as the interpreted form of the relation Θ_δ,

$$\widetilde{\Theta}_\delta := \left\{ \langle \mathbf{x}, \mathbf{y} \rangle \in (\mathrm{RV}_\delta^+)^2 \mid \exists z \in \mathcal{O} \ \left(\pi_\delta(z) = \vartheta(\mathbf{x}) \wedge \mathrm{rv}_\delta(z) = \mathbf{y} \right) \right\},$$

are definable in RV_δ.

Beginning with $\widetilde{\Theta}_\delta$, it happens in fact that

$$\widetilde{\Theta}_\delta(\mathbf{x}, \mathbf{y}) \Leftrightarrow \mathbf{x} \sim \mathbf{y},$$

i.e. iff $\vartheta(\mathbf{x}) = \vartheta(\mathbf{y})$. This is because given $y \in \mathcal{O}$ with $\mathrm{rv}_\delta(y) = \mathbf{y}$,

$$\pi_\delta(y) = \vartheta(\mathbf{x}) \Leftrightarrow \vartheta(\mathrm{rv}_\delta(y)) = \vartheta(\mathbf{y}) = \vartheta(\mathbf{x}) \Leftrightarrow \mathbf{x} \sim \mathbf{y}.$$

Similarly, $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in M$ iff $\mathbf{xy} \sim \mathbf{z}$ follows from again taking $x, y, z \in \mathcal{O}$ with $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(y) = \mathbf{y}, \mathrm{rv}_\delta(z) = \mathbf{z}$ and noticing that

$$\pi_\delta(x)\pi_\delta(y) = \pi_\delta(z) \Leftrightarrow \pi_\delta(xy) = \pi_\delta(z) \Leftrightarrow \vartheta(\mathbf{xy}) = \vartheta(\mathbf{z}) \Leftrightarrow \mathbf{xy} \sim \mathbf{z}.$$

Finally, for addition we have $A = \oplus_\delta \cap \left(\mathrm{RV}_\delta^+\right)^3$. To see this, taking $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{RV}_\delta^+$, we have

$$\begin{aligned} \oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \quad &\Leftrightarrow \quad \exists x, y, z \in \mathcal{O} \left(\mathrm{rv}_\delta(x) = \mathbf{x} \wedge \mathrm{rv}_\delta(y) = \mathbf{y} \wedge \mathrm{rv}_\delta(z) = \mathbf{z} \wedge x + y = z \right) \\ &\Leftrightarrow \quad \exists x, y, z \in \mathcal{O} \left(\pi_\delta(x) + \pi_\delta(y) = \pi_\delta(z) \right) \\ &\Leftrightarrow \quad \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in A. \end{aligned}$$

□

The reverse is also true:

Proposition 3.2.2. RV_δ *is interpretable in* K_δ.

Proof. It is only necessary to prove that the relation $\oplus_\delta$ is definable on G_δ. Suppose for simplicity that $\mathbf{x} \neq \infty$ and $v(\mathbf{x}) \leq v(\mathbf{y})$. We claim that $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ if and only if

$$\exists \mathbf{x^{-1}} \in G_\delta \ \exists \bar{a}, \bar{b} \in \mathcal{O}_\delta \left(\mathbf{x^{-1}x} = \mathbf{1} \wedge \Theta_\delta(\bar{a}, \mathbf{x^{-1}y}) \wedge \Theta_\delta(\bar{b}, \mathbf{x^{-1}z}) \wedge \bar{1} + \bar{a} = \bar{b} \right).$$

If there are $x, y, z \in K$ with $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(y) = \mathbf{y}, \mathrm{rv}_\delta(z) = \mathbf{z}$ and $x + y = z$, then $1 + x^{-1}y = x^{-1}z$ gives $\bar{1} + \bar{a} = \bar{b}$ in the above. Conversely, if the formula holds, then taking $x^{-1}, a, b \in K$ such that $\mathrm{rv}_\delta(x^{-1}) = \mathbf{x^{-1}}$, $\pi_\delta(a) = \bar{a}$, and $\pi_\delta(b) = \bar{b}$, we find that $\mathrm{rv}_\delta(x) = \mathbf{x}, \mathrm{rv}_\delta(xa) = \mathbf{y}, \mathrm{rv}_\delta(xb) = \mathbf{z}$ witness $\oplus_\delta(\mathbf{x}, \mathbf{y}, \mathbf{z})$. ☐

It is evident that 3.2.1 and 3.2.2 in fact give a biinterpretation, since the interpretations are given by the identity map between RV_δ and G_δ. Thus we have proven that the leading term language of order δ is biinterpretable with the amc-structure of order δ, for all $\delta \geq 0$ in V. Moreover, this biinterpretation is field-quantifier-free: the formulas defining the maps and interpreted relations include no quantifiers over the field sort.

3.3 Quantifier Elimination

Quantifier elimination holds in algebraically closed valued fields (see Robinson [28]), but such a property is too much to expect in general henselian valued fields. As noted in Example 1.1.9, a henselian valued field can be constructed with arbitrary residue field and value group.

Instead, one hopes for a *relative* quantifier elimination. This would mean that all quantifiers can be eliminated down to those which quantify over a certain definable subset, or over a sort. In the case of valued fields, for example, it may be possible to eliminate field-sorted quantifiers by showing that they can be transformed into quantifiers over the residue field sort or the value group sort, or over the leading term structures. In fact, in henselian fields of characteristic 0 in the leading term language, such a relative quantifier elimination does indeed hold.

This was shown by Kuhlmann in [20], following a special case by Basarab [4]. Considering Proposition 3.2.1, and the absence of field-sorted quantifiers in the interpretation given there, Kuhlmann's result transfers immediately into our modified leading term language.

Proposition 3.3.1 (Kuhlmann [20]). *Let (K, v) be a henselian valued field of characteristic 0. The theory of $(K, \langle \mathrm{RV}_\delta \rangle_{\delta \in \Delta})$ admits elimination of field-sorted quantifiers, where*

$$\Delta = \begin{cases} \{0\} & \textit{if } \mathrm{char}(R) = 0 \\ \{v(p^n) \mid n \in \mathbb{N}\} & \textit{if } \mathrm{char}(R) = p > 0. \end{cases}$$

□

Much like in Proposition 3.1.8, if the field contains a cross section of the value group definable without field-sorted quantifiers, the 3-sorted language is biinterpretable with the leading term language, and the relative quantifier elimination carries over. However, a definable cross section will not often exist. Thus, while the 3-sorted language has the benefit of dealing with more familiar structures, the leading term language seems to be the more natural language for henselian fields, as this is where the quantifier elimination works.

Example 3.3.2. We give an example of a henselian field which does not admit quantifier elimination relative to R and V. Let $K = \mathbb{Q}((t))$. As a Hahn field, K is henselian.

We claim that the elements t^2 and $2t^2$ satisfy the same field-quantifier-free formulas (without parameters). Such a formula can be written as a boolean combination of formulas of the form

(i) $f(x) = g(x)$ or $f(x) \neq g(x)$, where $f(x), g(x) \in \mathbb{N}[x]$;

(ii) $\varphi(\mathrm{res}(f_1(x)), \ldots, \mathrm{res}(f_n(x)))$, where φ is a formula in the residue field sort and each $f_i \in \mathbb{N}[x]$;

(iii) or $\psi(v(f_1(x)), \ldots, v(f_n(x)))$, where ψ is a formula in the value group sort and each $f_i \in \mathbb{N}[x]$.

In the first case, since t^2 and $2t^2$ are transcendental over $\mathbb{Q}$, it is easily seen that $f(t^2) = g(t^2)$ if and only if $f(2t^2) = g(2t^2)$ if and only if $f(x) = g(x)$. In (ii), for each i $\operatorname{res}(f_i(t^2))$ and $\operatorname{res}(f_i(2t^2))$ are both simply the residue of the constant term of f_i, and therefore equal. Thus t^2 satisfies a formula of the form in (ii) if and only if $2t^2$ does.

For (iii), if $f(x) = \sum_{i=0}^{d} a_i x^i$ with each $a_i \in \mathbb{Q}$, and suppose k is the least integer such that $a_k \neq 0$. Then since $v(a_i(t^2)^i) = v(a_i(2t^2)^i) = v(a_i) + 2i = 2i$ if $a_i \neq 0$ (and $= \infty$ if $a_i = 0$), we have $v(f(t^2)) = v(f(2t^2)) = 2k$. So again, t^2 satisfies ψ if and only if $2t^2$ does.

Therefore, t^2 and $2t^2$ have the same field-quantifier-free type over $\emptyset$. However, they do not have the same type, since only t^2 is a square in $\mathbb{Q}((t))$. The same example can be adapted to $k((t^V))$ for any field k containing an element lacking an n^{th} root in k, proving also that such a field cannot contain a cross section of V definable over $\emptyset$ without field-sorted quantifiers.

Chapter 4

Definable Subsets of K

In [12], Holly proved a canonical form for definable subsets (in one variable) of an algebraically closed valued field (ACVF). Namely, each such set can be expressed uniquely as a finite union of non-trivially nested swiss cheeses. This was used in [13] to show that these one-dimensional sets are coded in ACVF in the 3-sorted valued field language augmented by extra sorts for the balls.

In this chapter, we use the decomposition of Chapter 2 and relative quantifier elimination from Chapter 3 to give an analog of these results in characteristic 0 henselian valued fields, relativized to the leading term structures. Again, we work in a henselian valued field (K, v) with $\mathrm{char}(K) = 0$.

4.1 A canonical form

As with the quantifier elimination, in attempting to describe the definable subsets of K it will be impossible to make any assumption on what is happening in the leading term structures. What is sought is a simple characterization of definable subsets of the field in terms of definable sets in the leading term structures.

Proposition 4.1.1. *Suppose $S \subseteq K$ is definable over A. Then there are $\alpha_1, \ldots, \alpha_k \in \mathrm{acl}(A)$ and a subset $D \subseteq \mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_k}$ definable over $\mathrm{acl}(A)$ such that*

$$S = \{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x - \alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x - \alpha_k)\rangle \in D\}.$$

As before, if $\mathrm{char}(R) = 0$*, we may take* $\delta_i = 0$ *for all* i*; if* $\mathrm{char}(R) = p > 0$*, then the* δ_i *can be taken among* $v(p^n)$ *for* $n \in \mathbb{N}$*.*

Proof. The elimination of field-sorted quantifiers from Proposition 3.3.1 implies that S is definable by a formula of the form

$$\varphi\left(\mathrm{rv}_{\delta_1}(f_1(x)), \ldots, \mathrm{rv}_{\delta_k}(f_k(x))\right) \tag{4.1}$$

with φ being a formula in the RV sorts and each f_i a polynomial with coefficients over A.

Applying the decomposition of Proposition 2.2.4, there are swiss cheeses $U_1, \ldots, U_m$ partitioning K, for each $i \leq k$ RV polynomials $t_{i1}, \ldots, t_{im}$ (over $\mathrm{acl}(A)$), and for each $i \leq k$ and $j \leq m$ field elements $\alpha_{ij} \in \mathrm{acl}(A)$ such that (4.1) is equivalent to

$$\bigvee_{j=1}^{m} \left(x \in U_j \ \wedge \varphi\left(t_{1j}[\mathrm{rv}_{\delta_{1j}}(x - \alpha_{1j})], \ldots, t_{kj}[\mathrm{rv}_{\delta_{kj}}(x - \alpha_{kj})]\right)\right)$$

(with each $\delta_{ij} = \delta_i + v(p^n)$, some $n \in \mathbb{N}$, or $\delta_{ij} = 0$ in residue characteristic 0).

The condition $x \in U_j$ is definable in RV with parameters of the form $\mathrm{rv}(x - \beta)$. Without loss of generality we consider β to be among the α_{ij} and let ψ_j be the RV formula expressing

$$\psi_j(\mathbf{x}_1, \ldots, \mathbf{x}_k) \Longleftrightarrow x \in U_j \ \wedge \varphi\left(t_{1j}[\mathbf{x}_1], \ldots, t_{kj}[\mathbf{x}_k]\right).$$

For each $i \leq k$ define $\gamma_i := \max_{j \leq m}\{\delta_{ij}\}$. Since every $t_{ij}[\mathrm{rv}_{\delta_{ij}}(x - \alpha_{ij})]$ can be computed as $t_{ij}[\mathrm{rv}_{\gamma_i \to \delta_{ij}}(\mathrm{rv}_{\gamma_i}(x - \alpha_{ij}))]$, it may without loss of generality be assumed that $\delta_{ij} = \gamma_i$ for all i, j. Thus each ψ_j is a formula over $\mathrm{RV}_{\gamma_1} \times \ldots \times \mathrm{RV}_{\gamma_k}$.

Finally, letting χ be the formula $\bigvee \psi_j$ and D be the set in $\mathrm{RV}_{\gamma_1} \times \ldots \times \mathrm{RV}_{\gamma_k}$ defined by χ, we have

$$S = \{x \in K \mid \langle \mathrm{rv}_{\gamma_1}(x - \alpha_1), \ldots, \mathrm{rv}_{\gamma_k}(x - \alpha_k)\rangle \in D\}$$

as required. □

Holly's swiss cheeses in algebraically closed valued fields arise as boolean combinations of balls. This can be seen as the combination of a pullback of a finite set (from the residue field) and an interval (the value group). It is a consequence of strong minimality and o-minimality that these are all the sets definable in the residue field and value group. Again, it is a necessary byproduct of the relativity in the quantifier elimination in the henselian setting that we must allow for pullbacks of arbitrary definable sets D in the leading term structures.

The pullback of an interval in the value group itself will produce a ball (or, more accurately, an annulus) around 0. Shifting to balls centered elsewhere can be taken as analogous to our linear shifting by $\langle \alpha_1, \ldots, \alpha_k \rangle$.

4.2 Imaginaries

The main theorem of [13] gives a first step towards elimination of imaginaries for ACVF using the canonical form theorem of [12]. With 4.1.1 we may also prove a general henselian version. First, some definitions are needed.

Definition 4.2.1. Let T be a first-order theory, possibly in a multisorted language. T admits *elimination of imaginaries* if, for every model $M \models T$, $\emptyset$-definable set $S \subseteq M^n$, and $\emptyset$-definable equivalence relation $\sim$ on S, there is an $m \in \mathbb{N}$ and $\emptyset$-definable function $f : S \to M^m$ such that for all $a, b \in S$, $a \sim b$ if and only if $f(a) = f(b)$.

Thus $f(a)$ can act as a proxy for the equivalence class of a under $\sim$. These equivalence classes are the so-called 'imaginaries'.

Like the morleyization process for quantifier elimination, it is always possible to force a theory to eliminate imaginaries by expanding the language. This is done by adding, for each $\emptyset$-definable equivalence relation $\sim$, a new sort $\widetilde{S}$ consisting of the equivalence classes of $\sim$ as well as a new function symbol f which maps an element to its equivalence class in $\widetilde{S}$. The resulting theory is denoted T^{eq}.

For theories which fail to eliminate imaginaries, it can be useful to find some intermediate, optimal expansion lying between T and T^{eq} which suffices. This was done, for example, for ACVF in Haskell, Hrushovski, and Macpherson [10]. They found that algebraically closed valued fields eliminate imaginaries in the 3-sorted language augmented by new sorts made up of definable submodules of K^n over $\mathcal{O}$ and their cosets (the 'torsors'). Since the balls centered around 0 constitute $\mathcal{O}$-submodules of K, this is presented as a generalization of Holly's theorems to more dimensions.

From another perspective, elimination of imaginaries gives a way of treating definable sets as elements. Given a formula $\varphi(\bar{x}, \bar{y})$, define

$$\bar{a} \sim \bar{b} \text{ iff } \forall x \left(\varphi(\bar{x}, \bar{a}) \leftrightarrow \varphi(\bar{x}, \bar{b})\right)$$

i.e. if $\varphi(\bar{x}, \bar{a})$ and $\varphi(\bar{x}, \bar{b})$ define the same set. This set can then be identified with the equivalence class of $\bar{a}$ under $\sim$.

This motivates the following definition and proposition.

Definition 4.2.2. Let $S \subseteq M^n$ be a definable set of a model M. S is *coded* in M if there exist a formula $\varphi(\bar{x}, \bar{y})$ and a tuple $\bar{a}$ in M such that for all $\bar{b}$, $\varphi(\bar{x}, \bar{b})$ defines S if and only if $\bar{b} = \bar{a}$.

Poizat [24] proved that coding definable sets is equivalent to eliminating imaginaries:

Proposition 4.2.3. *Suppose T is a theory with at least two distinct definable elements (such as any theory of fields). T eliminates imaginaries if and only if for every $M \models T$ and definable $S \subseteq M^n$, S is coded in M.*

Returning to henselian valued fields, therefore, Proposition 4.1.1 suggests an expansion of the leading term language in which all definable subsets of K (in one variable) are coded. To do this, it must be assumed that all definable subsets in the leading term sorts are already coded. Thus, we begin with the language $\left(K, \langle \mathrm{RV}_\delta \rangle^{\mathrm{eq}}_{\delta \in \Delta}\right)$. A code for the definable $D \subseteq \mathrm{RV}^n$ will be denoted $\ulcorner D \urcorner$.

To be precise, by $\left(K, \langle \mathrm{RV}_\delta \rangle^{\mathrm{eq}}_{\delta \in \Delta}\right)$ is meant the model expanded to the language adding, for each formula $\varphi(\mathbf{x}, \mathbf{y})$ over the leading term sorts, a new sort S_φ for the equivalence classes of the relation

$$\mathbf{a} \sim \mathbf{b} \text{ iff } \forall \mathbf{x} \left(\varphi(\mathbf{x}, \mathbf{a}) \leftrightarrow \varphi(\mathbf{x}, \mathbf{b})\right)$$

and with a new function f_φ sending $\mathbf{a}$ to its equivalence class in S_φ.

Now, 4.1.1 showed that every definable subset of K is of the form

$$\{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x-\alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x-\alpha_k)\rangle \in D\}$$

for some definable D in the leading term sorts. We wish to give a language in which these sets are coded. Given the formula φ defining D, define a new equivalence relation $\sim_\varphi$ on $K^k \times S_\varphi$ by

$$\langle \alpha_1, \ldots, \alpha_k, \ulcorner D \urcorner\rangle \sim_\varphi \langle \beta_1, \ldots, \beta_k, \ulcorner E \urcorner\rangle \text{ iff}$$

$$\forall x\, (\langle \mathrm{rv}_{\delta_1}(x-\alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x-\alpha_k)\rangle \in D \leftrightarrow \langle \mathrm{rv}_{\delta_1}(x-\beta_1), \ldots, \mathrm{rv}_{\delta_k}(x-\beta_k)\rangle \in E)$$

and add as before a new sort T_φ consisting of the equivalence classes of $\sim_\varphi$, and a new function $g_\varphi : K^k \times S_\varphi \to T_\varphi$ sending $\langle \alpha_1, \ldots, \alpha_k, \ulcorner D \urcorner\rangle$ to its equivalence class.

Let $\mathcal{L}$ be the expansion of the leading term language with all these new sorts and coding functions added for every RV-formula φ.

Proposition 4.2.4. *Let $M = \left(K, \langle \mathrm{RV}_\delta\rangle^{eq}_{\delta\in\Delta}, \langle T_\varphi\rangle\right)$ be a henselian valued field of characteristic 0 expanded to $\mathcal{L}$ as described above. (Recall that $\Delta = \{0\}$ for $\mathrm{char}(R) = 0$, $\Delta = \{v(p^n) \mid n \in \mathbb{N}\}$ for $\mathrm{char}(R) = p$.) Every subset $S \subseteq K$ of the field is coded in M.*

Proof. Most of the work has already been done by 4.1.1, what remains being only a matter of making sense of the formalism. In particular, the form of definable sets from 4.1.1 still holds in the expanded language, since the relations $\sim_\varphi$ are all definable by equations of the same form.

Given a definable $S \subseteq K$, therefore, suppose

$$S = \{x \in K \mid \langle \mathrm{rv}_{\delta_1}(x-\alpha_1), \ldots, \mathrm{rv}_{\delta_k}(x-\alpha_k)\rangle \in D\}\,.$$

Fix the formula $\varphi(\mathbf{x};\mathbf{a})$ defining D. Now letting $\mathbf{c} = g_\varphi(\alpha_1, \ldots, \alpha_k, f_\varphi(\mathbf{a}))$, S is defined by the formula

$$\psi(x, \mathbf{c}) :=$$

$$\forall y_1, \ldots, y_k, \mathbf{y}\, (g_\varphi(y_1, \ldots, y_k, f_\varphi(\mathbf{y})) = \mathbf{c} \leftrightarrow \varphi(\mathrm{rv}_{\delta_1}(x-y_1), \ldots, \mathrm{rv}_{\delta_k}(x-y_k); \mathbf{y}))$$

and moreover, $\mathbf{c}$ is the *only* element of T_φ for which $\psi(x, \mathbf{c})$ defines S.

Thus S is coded in M via ψ and $\mathbf{c}$. $\square$

As with Holly's theorem, it is hoped that this one-dimensional case could form a foundation for a full elimination of imaginaries for characteristic 0 henselian fields. Just as a key insight in [10] was the identification of the definable $\mathcal{O}$-modules and torsors as the proper generalization of the balls to more dimensions, a key issue here is in finding a suitable many-dimensional version of the sets in 4.1.1.

In fact, it is an immediate consequence of quantifier elimination that definable subsets of K^n take the form

$$\{\langle x_1, \ldots, x_n\rangle \in K \mid \langle \mathrm{rv}_{\delta_1}(f_1(\bar{x})), \ldots, \mathrm{rv}_{\delta_k}(f_k(\bar{x}))\rangle \in E\} \tag{4.2}$$

where E is definable in $\mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_k}$ and each $f_i \in K[x_1, \ldots, x_n]$.

One could then obtain a trivial elimination of imaginaries by including the sets (4.2) as new sorts. An approach towards a more satisfying solution of the elimination of imaginaries problem may be to give a necessary and sufficient subclass of the polynomials f_i.

For example, one could hope to show that every definable set can be coded in terms of sets of the form (4.2) with the f_i being affine transformations of K^n. Though a counterexample has not been found, this seems overly optimistic.

Chapter 5

Relative Decidability

As in [7], the swiss cheese decomposition of Chapter 2 suggests a decision procedure relative to the leading term structures. Working over the p-adics, Cohen obtains a decision procedure for $\mathbb{Q}_p$ using the fact that the residue field (a finite field) and value group (Presburger arithmetic) are themselves decidable.

Again, such a result will not hold generally in henselian valued fields due to the lack of control over the residue field or value group. However, as with Kuhlmann's quantifier elimination, we can look for a relative procedure which reduces questions in the field to questions about the leading term structures. This is a relative decision procedure in the sense that if the leading term structures are themselves decidable, then the whole valued field is decidable; or alternatively, if we allow ourselves an oracle for the leading term structures, then we can construct a decision procedure for the whole field.

The proof relies on an effective relative quantifier elimination procedure (as opposed to Kuhlmann's, which does not give a clear algorithm). At present, it works only in residue characteristic 0. Nevertheless, it seems likely that a similar proof could be adapted for the mixed characteristic case, and we show precisely where the obstacle remains.

5.1 Quantifier elimination revisited

The first step in the quantifier elimination comes from deciding questions about when certain finite sets of balls have a non-empty intersection.

Proposition 5.1.1. *Let $z_i, a_i \in K$, $0 \leq \delta_i \in V$ for $i \leq n$. The formula*

$$\exists x \left(\bigwedge_{i \leq n} \mathrm{rv}_{\delta_i}(z_i) = \mathrm{rv}_{\delta_i}(x - a_i) \right) \tag{5.1}$$

is equivalent to a formula with no field-sorted quantifiers over the parameters $\mathrm{rv}_{\delta_i}(z_i)$, $\mathrm{rv}_{\delta_i}(a_i - a_j)$, and δ_i (or, more precisely, an element of value δ_i).

Proof. Notice that the set of x satisfying $\mathrm{rv}_{\delta_i}(z_i) = \mathrm{rv}_{\delta_i}(x - a_i)$ is in fact equal to the open ball $B_i := B_{>v(z_i)+\delta_i}(z_i + a_i)$. So what is sought is a means of testing for nonemptiness of the intersection of the balls B_i. Since finitely many balls having pairwise nonempty intersections implies a nonempty intersection, it will be sufficient to do so for the intersection of two balls. Thus we may assume $n = 2$.

Let us assume also that $v(z_1) + \delta_1 \leq v(z_2) + \delta_2$. This implies that $B_1 \cap B_2 \neq \emptyset$ iff $B_1 \supseteq B_2$ iff $z_2 + a_2 \in B_1$ iff $v(z_1 + a_1 - z_2 - a_2) > v(z_1) + \delta_1$.

- *Case 1:* $v(z_1) \leq v(a_1 - a_2)$, $v(z_1) \leq v(z_2)$, and $\delta_1 \leq \delta_2$.

  Then, by Proposition 1.3.8, $v(z_1 + a_1 - z_2 - a_2) > v(z_1) + \delta_1$ is equivalent to

  $$\exists \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{RV}_{\delta_1} \left(\begin{array}{c} v(\mathbf{w}_1) \neq v(\mathbf{w}_2) \wedge \\ \mathrm{rv}_{\delta_1}(z_1) - \mathrm{rv}_{\delta_1}(z_2) + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathbf{w}_1 \wedge \\ \mathrm{rv}_{\delta_1}(z_1) - \mathrm{rv}_{\delta_1}(z_2) + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathbf{w}_2 \end{array} \right)$$

  since the sum in RV_{δ_1} at least determines the valuation except when

  $$v(z_1 - z_2 + a_1 - a_2) > \min\{v(z_1), v(z_2), v(a_1 - a_2)\} + \delta_1 = v(z_1) + \delta_1.$$

- *Case 2:* $v(z_1) \leq v(a_1 - a_2)$, $v(z_1) \leq v(z_2)$, and $\delta_1 > \delta_2$.

  This time, although $\mathrm{rv}_{\delta_1}(z_2)$ is no longer uniquely determined from $\mathrm{rv}_{\delta_2}(z_2)$, $v(z_1 + a_1 - z_2 - a_2) > v(z_1) + \delta_1$ is equivalent to

  $$\forall \mathbf{u} \in \mathrm{RV}_{\delta_1} \, \exists \mathbf{w}_1, \mathbf{w}_2 \in \mathrm{RV}_{\delta_1}$$

$$\left(\mathrm{rv}_{\delta_2}(\mathbf{u}) = \mathrm{rv}_{\delta_2}(z_2) \to \left(\begin{array}{c} v(\mathbf{w}_1) \neq v(\mathbf{w}_2) \wedge \\ \mathrm{rv}_{\delta_1}(z_1) - \mathbf{u} + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathbf{w}_1 \wedge \\ \mathrm{rv}_{\delta_1}(z_1) - \mathbf{u} + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathbf{w}_2 \end{array}\right)\right)\right)$$

because $\mathrm{rv}_{\delta_2}(\mathbf{u}) = \mathrm{rv}_{\delta_2}(z_2)$ implies that $\mathrm{rv}_{\delta_1}(z_1) - \mathbf{u} = \mathrm{rv}_{\delta_1}(z_1) - \mathrm{rv}_{\delta_1}(z_2)$.

To see this, let $\mathbf{u} = \mathrm{rv}_{\delta_1}(u)$ and note that the inequality $v(z_1) < v(z_2)$ is in fact strict. Thus $\mathrm{rv}_{\delta_1}(z_1) - \mathbf{u} = \mathrm{rv}_{\delta_1}(z_1 - u)$ and $\mathrm{rv}_{\delta_1}(z_1) - \mathrm{rv}_{\delta_1}(z_2) = \mathrm{rv}_{\delta_1}(z_1 - z_2)$ as well-defined sums. Now

$$v((z_1 - u) - (z_1 - z_2)) = v(z_2 - u) > v(z_2) + \delta_2 \geq v(z_1) + \delta_1$$

by $\mathrm{rv}_{\delta_2}(u) = \mathrm{rv}_{\delta_2}(z_2)$.

Now argue as in Case 1.

- *Case 3:* $v(z_1) \leq v(a_1 - a_2)$ and $v(z_2) < v(z_1)$.

  This implies $v(z_1 + a_1 - z_2 - a_2) = v(z_2) < v(z_1) + \delta_1$, so this case is trivial.

- *Case 4:* $v(a_1 - a_2) < v(z_1)$.

  In this case, $\mathrm{rv}_{\delta_1}(z_1) + \mathrm{rv}_{\delta_1}(a_1 - a_2)$ is well-defined. Then

  $$\exists x\, (\mathrm{rv}_{\delta_1}(z_1) = \mathrm{rv}_{\delta_1}(x - a_1) \wedge \mathrm{rv}_{\delta_2}(z_2) = \mathrm{rv}_{\delta_2}(x - a_2))$$

  holds if and only if

  $$\exists x\, (\mathrm{rv}_{\delta_1}(z_1) + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathrm{rv}_{\delta_1}(x - a_2) \wedge \mathrm{rv}_{\delta_2}(z_2) = \mathrm{rv}_{\delta_2}(x - a_2))\,.$$

  If $\delta_1 \leq \delta_2$ this is equivalent to

  $$\mathrm{rv}_{\delta_1}(z_1) + \mathrm{rv}_{\delta_1}(a_1 - a_2) = \mathrm{rv}_{\delta_1}\left(\mathrm{rv}_{\delta_2}(z_2)\right)$$

  (witnessed when the above holds by $x = z_2 + a_2$), while if $\delta_2 < \delta_1$ it is equivalent to

  $$\mathrm{rv}_{\delta_2}\left(\mathrm{rv}_{\delta_1}(z_1) + \mathrm{rv}_{\delta_1}(a_1 - a_2)\right) = \mathrm{rv}_{\delta_2}(z_2)$$

  (witnessed by $x = z_1 + a_1$).

The desired formula will then be the disjunction over all these cases. □

In fact, we will need the above result to apply more generally to formulas involving the leading terms of polynomials linear in x.

Proposition 5.1.2. *Let $z_i, a_i, b_i \in K$ with $a_i \neq 0$. The formula*

$$\exists x \left(\bigwedge_{i \leq n} \mathrm{rv}_{\delta_i}(z_i) = \mathrm{rv}_{\delta_i}(a_i x - b_i) \right) \tag{5.2}$$

is equivalent to a formula with no field-sorted quantifiers over parameters $\mathrm{rv}_{\delta_i}(z_i)$, $\mathrm{rv}_{\delta_i}(a_j)$, $\mathrm{rv}_{\delta_i}(a_i b_j - a_j b_i)$, and δ_i.

Proof. This is easily adapted from 3.2.1. By factoring out $\mathrm{rv}(a_i)$ in (5.2) to get $\mathrm{rv}_{\delta_i}(z_i)\,\mathrm{rv}_{\delta_i}(a_i)^{-1} = \mathrm{rv}_{\delta_i}(x - b_i/a_i)$, 3.2.1 gives an equivalent field-quantifier-free formula over parameters $\mathrm{rv}_{\delta_i}(z_i)\,\mathrm{rv}_{\delta_i}(a_i)^{-1}$, $\mathrm{rv}_{\delta_i}(b_j/a_j - b_i/a_i)$, and δ_i. Factoring $\mathrm{rv}_{\delta_i}(a_i)$ and $\mathrm{rv}_{\delta_i}(a_i)\,\mathrm{rv}_{\delta_i}(a_j)$ back into the first two leading terms gives $\mathrm{rv}_{\delta_i}(z_i)$ and $\mathrm{rv}_{\delta_i}(a_i b_j - a_j b_i)$, as required. □

Proposition 5.1.2 forms the basis for an induction on the maximum degree of a polynomial appearing as a leading term. As noted above, the proof as a whole works only in residue characteristic 0, since we must eventually assume that each $\delta_i = 0$, but it is carried out as far as possible without this assumption.

The relative quantifier elimination essentially uses the linearization of the leading terms of polynomials to push questions about the existence of field elements into the leading term structures.

One feature of this method is that we need not make any assumptions on the formula in the RV structures. Consequently, we may allow any additional structure (such as a cross section, or an expansion to $\mathrm{RV}^{\mathrm{eq}}$) on the leading terms. The important point is that the field sort carries only the usual ring language and the map(s) rv_δ.

The basic situation, therefore, would be a two-sorted structure (K, RV) in residue characteristic 0, and a many-sorted structure $(K, \mathrm{RV}_0, \mathrm{RV}_{v(p)}, \mathrm{RV}_{v(p^2)}, \ldots)$ in residue characteristic $p > 0$. In full generality, however, the language can include any expansion on the RV part of these basic languages.

Proposition 5.1.3. *Let T be the theory of a characteristic 0 henselian field with* $\operatorname{char}(R) = 0$[1] *in a language of the kind described above. Then T eliminates field-sorted quantifiers.*

Proof. We begin with an existential formula of the form

$$\exists x \in K \left(\varphi\left(\operatorname{rv}_{\delta_1}(f_1(x, \bar{u})), \ldots, \operatorname{rv}_{\delta_n}(f_n(x, \bar{u}))\right)\right) \tag{5.3}$$

where φ is some predicate definable (with RV-sorted parameters) in $\mathrm{RV}_{\delta_1} \times \ldots \times \mathrm{RV}_{\delta_n}$, the f_i are polynomials over K, and all field-sorted free variables are among $\bar{u}$. It suffices to show that this is equivalent to a field-quantifier-free formula.

First of all, if each f_i is linear in x, rewrite (5.3) as

$$\exists \mathbf{z_i} \in \mathrm{RV}_{\delta_i} \left(\varphi\left(\mathbf{z_1}, \ldots, \mathbf{z_n}\right) \;\wedge\; \exists x \in K \left(\bigwedge_{i \le n} \mathbf{z_i} = \operatorname{rv}_{\delta_i}(f_i(x, \bar{u}))\right)\right). \tag{5.4}$$

Now Proposition 3.2.2 applies to eliminate the quantifier $\exists x$.

In general, we use an induction on $\max_{i \le n} \{\deg(f_i(x))\}$ to reduce each f_i to linear terms. Proposition 2.2.4 gives a partition $K = \dot{\bigcup} S_j$ and roots $\lambda_1, \ldots, \lambda_m$ of certain polynomials $g_1, \ldots, g_m$ which are derivatives of the f_i (in particular of degree $\le \max_{i \le n} \{\deg(f_i(x))\}$), such that for all $x \in S_j$ and $i \le n$ either

(i) $\operatorname{rv}_{\delta_i}(f_i(x))$ can be computed in RV_{δ_i} as a well-defined polynomial function of $\operatorname{rv}_{\delta_i}(x - \lambda_{ij})$ ($\lambda_{ij} \in \{\lambda_1, \ldots, \lambda_m\}$); or,

(ii) $\operatorname{rv}_{\delta_i}(f_i(x))$ can be computed as the image in RV_{δ_i} of a well-defined polynomial function of $\operatorname{rv}_{\delta_i + v(q)}(x - \lambda_{ij})$, q a power of p (only in case $\operatorname{char}(R) = p > 0$).

The roots $\lambda_1, \ldots, \lambda_m$ also serve as centers of the balls comprising the swiss cheeses S_j.

In this way, the formula in (5.3) is equivalent to one of the form

$$\begin{gathered} \exists x, y_1, \ldots, y_m \in K \; \bigwedge_i g_i(y_i) = 0 \;\wedge \\ \bigvee_j \varphi_j\left(\operatorname{rv}_{\delta_{k_{1j}}}(x - y_{1j}), \ldots, \operatorname{rv}_{\delta_{k_{nj}}}(x - y_{nj})\right). \end{gathered} \tag{5.5}$$

---

[1] As mentioned above, however, this assumption is avoided in the proof as far as possible.

Specifically, φ_j will express that $x \in S_j$, that there is a term $t_{ij}[\mathrm{rv}_{\delta_{k_{ij}}}(x - y_{ij})]$ which is well-defined for each i, and that φ holds with $t_{ij}[\mathrm{rv}_{\delta_{k_{ij}}}(x - y_{ij})]$ substituted for f_i. Here the y_{ij} are among $y_1, \ldots, y_m$ and the $\delta_{k_{ij}}$ are either δ_i or (in residue characteristic p only) $\delta_i + v(q)$.

In (5.5) the bound variable x occurs only linearly, so it can be eliminated as shown above. This produces an equivalent formula in the form

$$\exists y_1, \ldots, y_m \in K \left(\left(\bigwedge_i g_i(y_i) = 0 \right) \wedge \psi \left(\mathrm{rv}_{\gamma_1}(h_1(\bar{y}, \bar{u})), \ldots, \mathrm{rv}_{\gamma_\ell}(h_\ell(\bar{y}, \bar{u})) \right) \right) \tag{5.6}$$

with $h_1, \ldots, h_\ell$ being polynomials and ψ an RV formula. So it remains to show that the quantifiers $\exists y_1, \ldots, y_m$ can be eliminated in such a formula.

In fact we may do so one at a time, so writing y for y_1, g for g_1, and suppressing y_2 through y_m (i.e., treating them as free variables in the subformula) our goal becomes to eliminate the quantifier in a formula of the form

$$\exists y \in K \left(g(y) = 0 \wedge \psi \left(\mathrm{rv}_{\gamma_1}(h_1(y, \bar{u})), \ldots, \mathrm{rv}_{\gamma_\ell}(h_\ell(y, \bar{u})) \right) \right) \tag{5.7}$$

where $g(y)$ is a polynomial of degree $\leq \max_{i \leq n} \{\deg(f_i(x))\}$.

Moreover, given $g(y) = 0$ each $h_j(y)$ may be replaced with its remainder on division by $g(y)$ (in applying the euclidean algorithm, it will be necessary to multiply through by powers of the leading coefficient of g). Therefore it can be assumed that

$$\deg(h_j(y)) < \deg(g(y)) \leq \max_{i \leq n} \{\deg(f_i(x))\}$$

for each $j \leq \ell$. If the latter inequality is strict, of course, the induction hypothesis finishes the job.

Otherwise, if we have equality, let us apply the decomposition of Proposition 2.2.4 a second time relative to $h_1(y), \ldots, h_\ell(y)$. The result is another formula equivalent to (5.7) taking the form

$$\exists z_1, \ldots, z_k \in K \left(\bigwedge_{i \leq k} s_i(z_i) = 0 \wedge \exists y \left(g(y) = 0 \wedge \chi \left(\mathrm{rv}_{\eta_1}(y - z_1), \ldots, \mathrm{rv}_{\eta_k}(y - z_k) \right) \right) \right) \tag{5.8}$$

with $\deg(s_i(z_i)) \leq \max_{j \leq \ell} \{\deg(h_j(y))\} < \max_{j \leq n} \{\deg(f_j(x))\}$ for every i. Now, it will suffice to eliminate the quantifier $\exists y$ from the subformula

$$\exists y \left(g(y) = 0 \wedge \chi \left(\mathrm{rv}_{\eta_1}(y - z_1), \ldots, \mathrm{rv}_{\eta_k}(y - z_k)\right)\right) \tag{5.9}$$

since then we would be in the situation of (5.6) except now with the degrees of the $s_i(z_i)$ *strictly* less than $\max_{j \leq n} \{\deg(f_j(x))\}$.

From this point on, we need the assumption that $\mathrm{char}(R) = 0$*. So in particular the assumption will now be made that all* $\eta_i = 0$*.*

Suppose without loss of generality that $v(y - z_1) \geq v(y - z_i)$ for all $i \leq k$. Then for each i, $\mathrm{rv}(y - z_i) = \mathrm{rv}(y - z_1) + \mathrm{rv}(z_1 - z_i)$ is a well-defined sum. Thus $\chi \left(\mathrm{rv}(y - z_1), \ldots, \mathrm{rv}(y - z_k)\right)$ depends only on $\mathrm{rv}(y - z_1)$ and the parameters $\mathrm{rv}(z_1 - z_i)$, and in this case we may think of (5.9) as

$$\exists y \left(g(y) = 0 \wedge \chi' \left(\mathrm{rv}(y - z_1)\right)\right). \tag{5.10}$$

If we can eliminate the $\exists y$ in this formula, then by taking the disjunction over the possible cases of which $v(y - z_i)$ is largest, we will be done.

Now, write $g(y) = \sum_{i=0}^{d} a_i (y - z_1)^i$ (the a_i here will depend on $z_1, \ldots, z_k$, but as usual we suppress mention of the free variables and parameters wherever possible). If $a_0 = 0$, then z_1 is a root of g and we may check whether χ' holds on $\mathrm{rv}(z_1 - z_1) = \infty$.

Otherwise, if $g(y) = 0$ shares a common factor with $g^{(n)}$ for some $n \geq 1$ (but $n \leq d$), then we may factor to write $g(y) = g^{(n)}(y)\tilde{g}(y)$ and reduce (5.10) to

$$\exists y \left(\left(g^{(n)}(y) = 0 \wedge \chi' \left(\mathrm{rv}(y - z_1)\right)\right) \vee \left(\tilde{g}(y) = 0 \wedge \chi' \left(\mathrm{rv}(y - z_1)\right)\right)\right).$$

Since both $g^{(n)}$ and $\tilde{g}$ have degree strictly less than $\deg(g)$, we may apply the induction to finish the proof. Thus we assume that g has no common factors with any of its derivatives.

Finally, if $g(y) = 0$ then g has a collision at y around z_1, and no root of g is also a root of any of $g', g'', \ldots, g^{(d)}$. Now, Proposition 2.1.3 implies that there is a λ and $n < d$ such that $g^{(n)}(\lambda) = 0$ and $\mathrm{rv}(\lambda - z_1) = \mathrm{rv}(y - z_1)$.

Consider the formula

$$\forall\lambda\exists y\left(\left(\bigvee_{1\leq i<d} g^{(i)}(\lambda)=0\right)\rightarrow \text{`}g \text{ has a collision at } y \text{ around } \lambda\text{'} \wedge \chi'(\mathrm{rv}(y-z_1))\right). \tag{5.11}$$

We claim that (5.11) is equivalent to (5.10). In fact, if g has a root at y and λ is a root of $g^{(i)}$ $(1 \leq i < d)$, then $g(\lambda) \neq 0$ and so the constant term of g recentered around λ is nonzero. This implies that g still has a collision at y around λ.

If, conversely, (5.11) holds, then Proposition 2.2.2 implies that there is a λ, which is a root of one of $g, g', g'', \ldots, g^{(d)}$ for which $\mathrm{rv}(y-z_1) = \mathrm{rv}(\lambda - z_1)$ and g does not have a collision at y around λ. Therefore, this λ must be a root of g itself. In other words, (5.11) implies that there is a root of λ of g for which $\mathrm{rv}(y-z_1) = \mathrm{rv}(\lambda - z_1)$, and χ' holds of $\mathrm{rv}(y-z_1)$. This shows that (5.10) and (5.11) are equivalent.

In (5.11), the quantifier $\exists y$ can be eliminated as in (5.4), since having a collision at y around λ is a definable condition on $\mathrm{rv}(y-\lambda)$, and thus y appears only linearly. Likewise, the quantifier $\forall\lambda$ can also be eliminated by the induction hypothesis, because $\deg(g^{(i)}) < \deg(g)$.

Taking the disjunction over all these cases, we have succeeded in eliminating the field-sorted quantifier in (5.9), and this finishes the proof. □

Since, looking back over the proof of Proposition 2.2.2, we have an effective algorithm for producing the swiss cheese decomposition, the above proof gives an effective algorithm for producing a field-quantifier-free formula from any formula in the leading term language. Assuming formulas in the leading term sorts are decidable, therefore, we may use this to devise a decision procedure for formulas over the valued field, and we have proved

Proposition 5.1.4. *The theory of a henselian valued field with* $\mathrm{char}(K) = \mathrm{char}(R) = 0$ *is decidable relative to an oracle for* RV*, or equivalently, as long as* RV *is decidable.*

□

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