A LABELLING OF THE FACES IN THE SHI ARRANGEMENT

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The Shi Arrangement

The n-dimensional Shi Arrangement $\mathcal{S}_n$ consists of the $n(n-1)$ hyperplanes in $\mathbb{R}^n$ defined by the equations

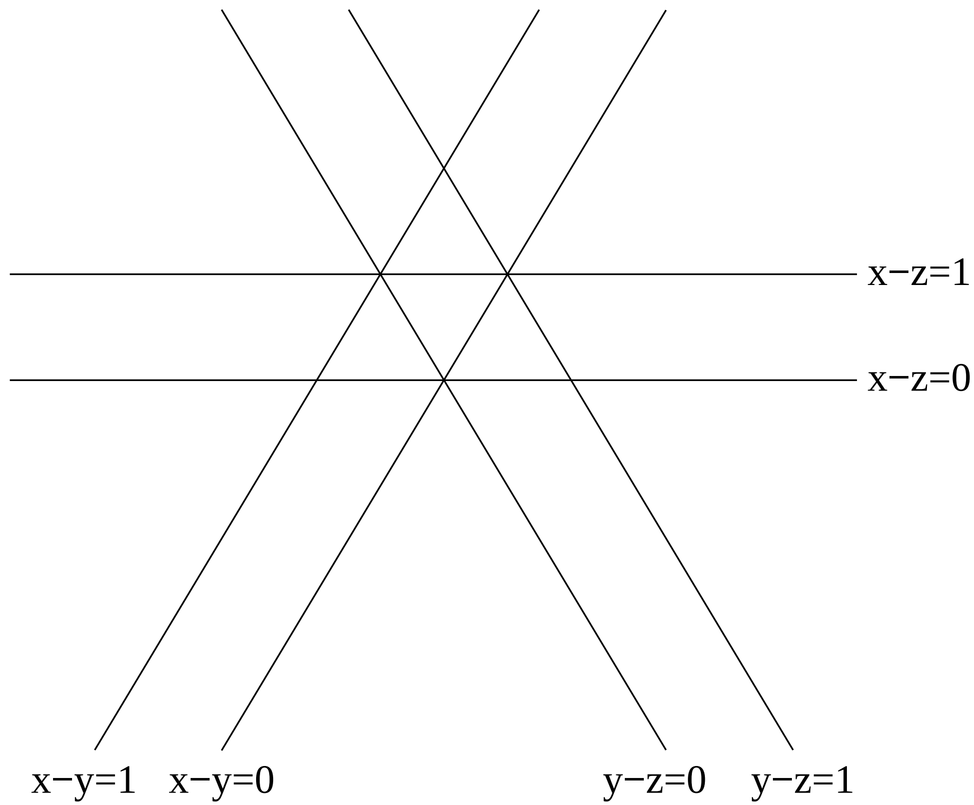
$$x_i - x_j = 0 \qquad x_i - x_j = 1$$

$$\text{for } 1 \leq i < j \leq n,$$

where x_i is the i-th coordinate of $\mathbb{R}^n$.

The Case n=3

Take x, y, z as the coordinates of $\mathbb{R}^3$. Notice that all planes in $\mathcal{S}_3$ are orthogonal to the plane $x + y + z = 0$. The next figure shows their intersection with this particular plane.



Parking Functions

Let $[n] = \{1, 2, \ldots, n\}$. A parking function of length n is a secuence $P = (P_1, P_2, \ldots, P_n) \in [n]^n$ such that if $Q_1 \leq Q_2 \leq \cdots \leq Q_n$ is the increasing rearrangement of the entries in P then $Q_i \leq i$.

n=2: $11, 12, 21.$

n=3: $111, 112, 113, 121, 122, 123, 131, 132, 211, 212, 213,$ $221, 231, 311, 312, 321.$

There are $(n+1)^{n-1}$ parking functions of length n.

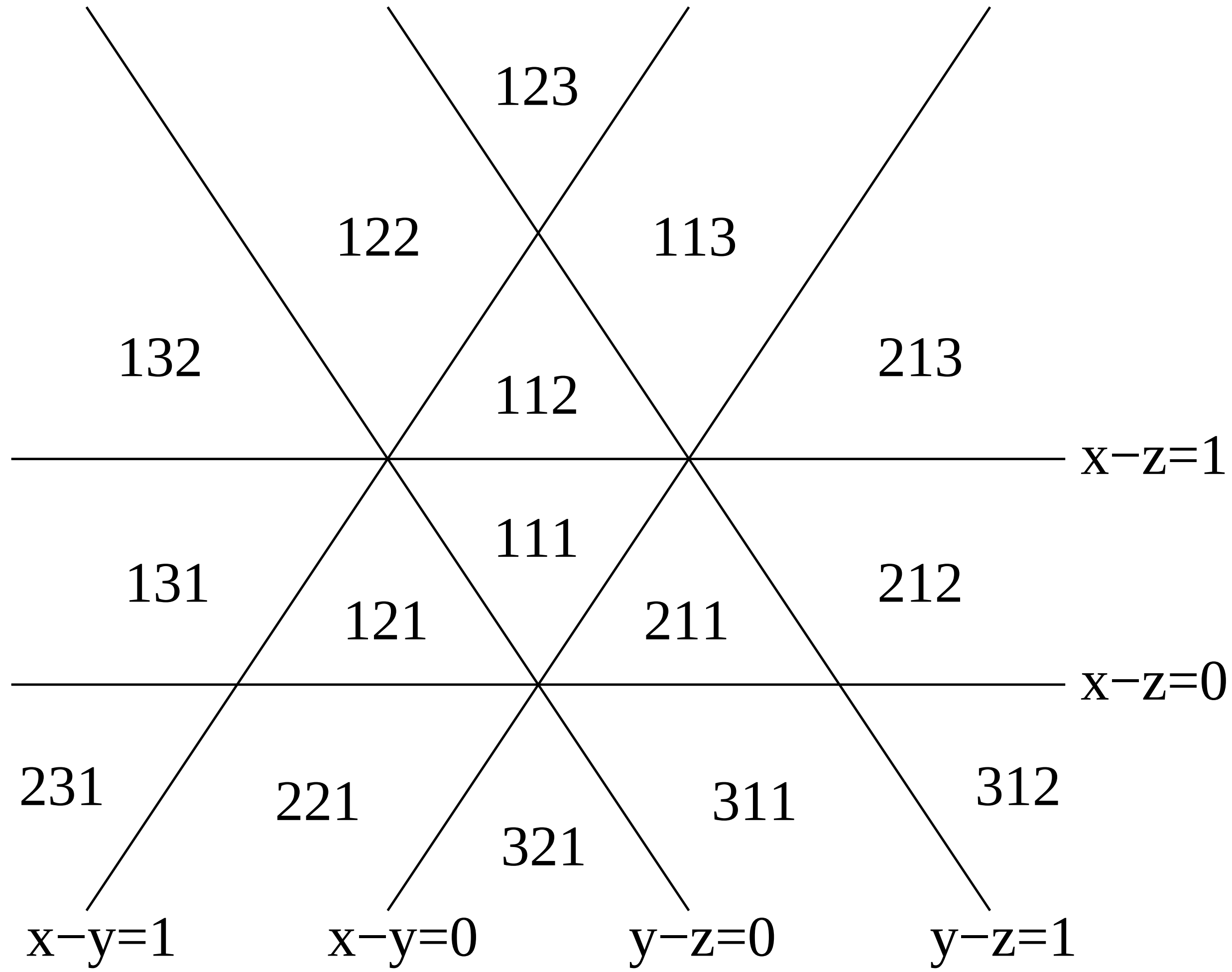
Regions and Parking Functions

$\mathcal{S}_n$ cuts $\mathbb{R}^n$ into several open connected components, called regions. There is a bijection between these regions and the parking functions of length n (Pak and Stanley):

- Consider the central region R_0 defined by $x_n < x_{n-1} < \ldots < x_1 < x_n + 1$. Assign it the label $\lambda(R_0) = (1, 1, \ldots, 1)$.

- Suppose region R is labelled. If the regions R' and R are separated only by the hyperplane $x_i - x_j = 0$ (with $i < j$), define $\lambda(R') = \lambda(R) + e_i$ (e_i is the i-th vector of the canonical basis). If the regions R' and R are separated only by the hyperplane $x_i - x_j = 1$ (with $i < j$), define $\lambda(R') = \lambda(R) + e_j$.

Bijection with n=3



Faces in the Shi Arrangement

Any nonempty intersection between the closure of a region and some (or none) hyperplanes in $\mathcal{S}_n$ is called a face.

The poset consisting of all the faces ordered by inclusion is called the face poset of $\mathcal{S}_n$.

Main Theorem - The Labelling of the Faces

Let $\mathcal{P}_n$ be the poset of parking functions of length n, with the order defined by $(P_1, P_2, \ldots, P_n) \leq (Q_1, Q_2, \ldots, Q_n)$ if $P_i \leq Q_i$ for all i.

Let F be a face of $\mathcal{S}_n$. Then the set of labels of the regions adjacent to F (that is, the regions that contain F in their closure) is a closed interval of $\mathcal{P}_n$. We will denote this interval by $\widehat{\lambda}(F)$.

The Labelling of the Faces

Corollary: Let $R_1, R_2, \ldots, R_k$ be some regions of $\mathcal{S}_n$, and let $P_1, P_2, \ldots, P_k$ be their labels. If there is no parking function P such that $P_i \leq P$ for all i, then $\bigcap_{i=1}^{k} \overline{R_i} = \emptyset$.

Notice that if F is a region of $\mathcal{S}_n$ then $\widehat{\lambda}(F) = \{\lambda(F)\}$, so $\widehat{\lambda}$ can be considered as an extension of the labelling λ.

Combinatorial Information of the Face Poset

Let F, G be two faces of $\mathcal{S}_n$. Then $F \subseteq G$ if and only if $\widehat{\lambda}(F) \supseteq \widehat{\lambda}(G)$. This shows that a characterization of all intervals appearing as labels of some face will give us a complete combinatorial description of the face poset of the Shi Arrangement.

Some Properties of the Labelling

Let F be a face of $\mathcal{S}_n$ and $\widehat{\lambda}(F) = [P, Q]$. Then

$$\dim(F) = \#\{i \in [n] \mid P_i = Q_i\}$$

and

$$\{Q_1 - P_1, Q_2 - P_2, \ldots, Q_n - P_n\} = \{0, 1, 2, \ldots, m\}$$

for some $m \in \mathbb{N}$.

This last statement restricts a lot the possible intervals of $\mathcal{P}_n$ appearing as labels of some face.

Possible Sizes

The set

$$\left\{\#\widehat{\lambda}(F) \mid \text{dim}(F) = k\right\}$$

is the set of all positive integers d such that $d = 2^{a_1}3^{a_2}\ldots(m+1)^{a_m}$ for some $m \in \mathbb{N}$, where $a_i > 0$ for all $i \leq m$, and $a_1 + a_2 + \ldots + a_m = n - k$.

This proposition is giving us some geometrical information about the Shi Arrangement.

Labels of 1-dimensional Faces

The interval $[P, Q]$ is the label of a 1-dimensional face if and only if the following statements hold:

- Q is a permutation of $[n]$.

- P is determined by Q in the following way. Denote $\Big(a_1, a_2, \ldots, a_n\Big) = \Big(Q^{-1}(1), Q^{-1}(2), \ldots, Q^{-1}(n)\Big)$, and let $0 = i_0 < i_1 < i_2 < \ldots < i_k = n$ be the numbers such that

  $$\{i_1, i_2, \ldots, i_{k-1}\} = \Big\{j \in [n] \mid a_j < a_{j+1}\Big\}.$$

  Then for all $r \in [n]$, if j is such that $i_j < r \leq i_{j+1}$ we have that

  $$P_{a_r} = i_{j-1} + \Big|\Big\{l \in [n] \mid i_{j-1} < l \leq i_j \text{ and } a_l > a_r\Big\}\Big| + 1,$$

  where $i_{-1} = 0$.

Counting 1-dimensional faces

Each region R such that $\lambda(R)$ is a permutation of $[n]$ contains a unique 1-dimensional face. Moreover, each 1-dimensional face is contained in a unique region R such that $\lambda(R)$ is a permutation of $[n]$.

Corollary: The number of 1-dimensional faces is $n!$.

Open Questions

Can these results be generalized to extended Shi Arrangements, or other arrangements?

Is there a simple characterization of the intervals appearing as labels of some face?

Can faces of dimension higher than 1 be counted in some similar way?