Valuations for matroid polytope subdivisions

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Outline

Matroids

Definition

A matroid M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a collection of subsets $\mathcal{B}$ of E satisfying:

- If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$.

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Example

If $k \leq n$ then the subsets of size k of $[n] = \{1, \ldots, n\}$ are the bases of a matroid, called the uniform matroid $U_{k,n}$.

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Example

Given $1 \leq s_1 < \cdots < s_r \leq n$, the sets $\{a_1 < \cdots < a_r\}$ such that $a_1 \leq s_1, \ldots, a_r \leq s_r$ are the bases of a matroid, called the Schubert matroid $SM_n(s_1, \ldots, s_r)$.

Matroid Polytopes

Definition

Given a matroid $M = ([n], \mathcal{B})$, its matroid polytope is

$$Q(M) = \mathrm{convex}\{e_B \,:\, B \in \mathcal{B}\},$$

where $e_B = \sum_{i \in B} e_i$.

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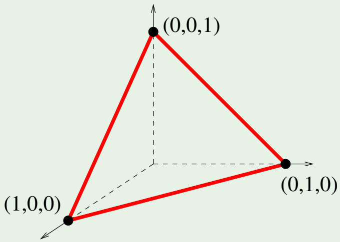
Example

If $M = U_{1,3}$ then $Q(M)$ is the triangle



A nice characterization

Theorem (Gelfand, Goresky, MacPherson, Serganova)

Let $\mathcal{B}$ be a collection of subsets of $[n]$ and let $Q(\mathcal{B}) = \mathrm{convex}\{e_B \,:\, B \in \mathcal{B}\}$. The following are equivalent:

1. *$\mathcal{B}$ is the collection of bases of a matroid.*
2. *Every edge of $Q(\mathcal{B})$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.*

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Corollary

Any face of a matroid polytope is itself a matroid polytope.

Matroid Subdivisions

Definition

A subdivision of a polytope P is a set of polytopes $S = \{P_1, \ldots, P_m\}$ whose vertices are vertices of P, such that

- $P_1 \cup \cdots \cup P_m = P$, and
- for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty then it is a proper face of both P_i and P_j.

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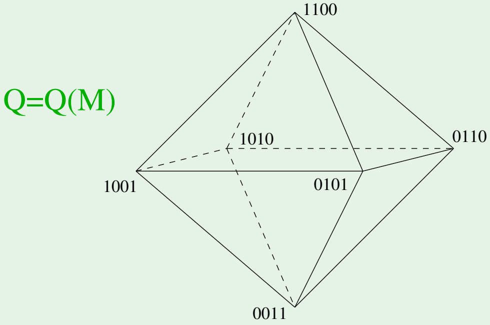
Notation

Given $A \subseteq [m]$ we will write $Q_A = \bigcap_{a \in A} Q_a$ (where $Q_\emptyset = Q$). We have that Q_A is the matroid polytope of a matroid M_A.

Matroid Subdivisions

Example

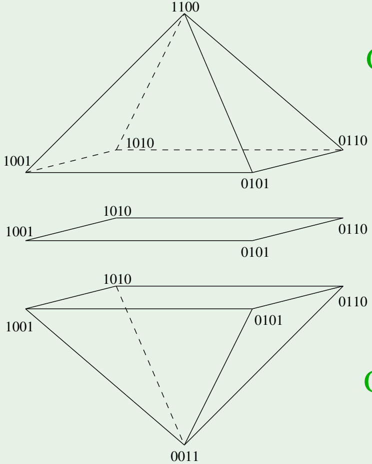
Let $M = U_{2,4}$. Its matroid polytope is



Matroid Subdivisions

Example

It can be subdivided into two pieces:



$Q_1 = Q(M_1)$

$Q_{12} = Q(M_{12})$

$Q_2 = Q(M_2)$

Why is this interesting?

These subdivisions have appeared prominently in different contexts:

- Compactifying the moduli space of hyperplane arrangements (Hacking, Keel and Tevelev; also Kapranov).
- Compactifying fine Schubert cells in the Grassmannian (Lafforgue).
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So, how can we get a handle on them?

Outline

Valuations

Definition

Let Mat $=$ Mat$_n$ denote the set of matroids with ground set $[n]$, and let G be some abelian group.

A function f : Mat $\to G$ is a valuation if for any subdivision of a matroid $M \in$ Mat into matroids $M_1, \ldots, M_m$ we have

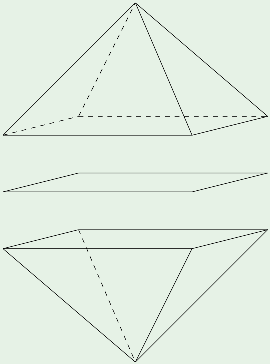
$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots ,$$

adopting the convention that $f(\emptyset) = 0$.

Valuations

Examples

- $f(M) = \mathrm{vol}(Q(M))$.



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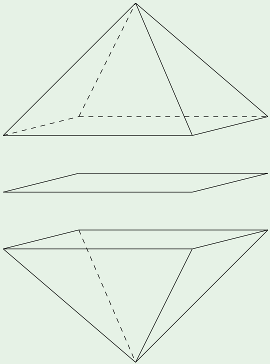

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- $f(M) = \mathrm{vol}(Q(M))$.
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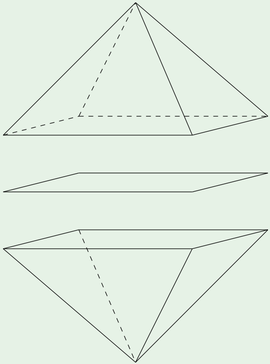
- $f(M) =$ Ehrhart polynomial of $Q(M)$.



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Indicator Functions

Definition

Given $X \subseteq \mathbb{R}^n$, let $i_X : \text{Mat} \to \mathbb{Z}$ be defined by

$$i_X(M) = \begin{cases} 1 & \text{if } Q(M) \cap X \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem

If $X \subseteq \mathbb{R}^n$ is convex, and is either open or closed, then i_X is a valuation.

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The Tutte Polynomial

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The Tutte polynomial of a matroid M is

$$T_M(x, y) = \sum_{A \subseteq [n]} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}.$$

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What can we say about its coefficients?

Basis Activities

Definition

Let B be a basis of $M = ([n], \mathcal{B})$.

- An element $i \in B$ is internally active over B if $i < j$ for all $j \notin B$ such that $B - i \cup j \in \mathcal{B}$.
- An element $i \notin B$ is externally active over B if $i < j$ for all $j \in B$ such that $B - j \cup i \in \mathcal{B}$.

Denote $I(B)$ and $E(B)$ be the sets of internally and externally active elements over B.

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Theorem (Crapo, Tutte)

The Tutte polynomial of M is

$$T_M(x, y) = \sum_{B \text{ basis of } M} x^{|I(B)|} y^{|E(B)|}.$$

It is a valuation!

Theorem (Speyer)

The Tutte polynomial is a valuation.

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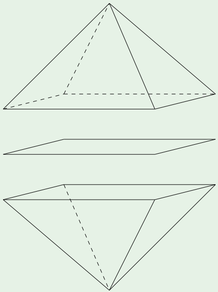
Example

- $T_M(x,y) = x^2 + y^2 + 2x + 2y$.
- $T_{M_1}(x,y) = x^2 + y^2 + xy + x + y$.
- $T_{M_2}(x,y) = x^2 + y^2 + xy + x + y$.
- $T_{M_{12}}(x,y) = x^2 + y^2 + 2xy$.



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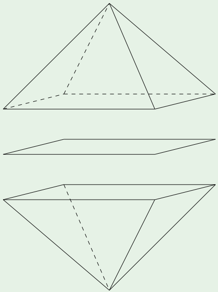

Are subset ranks or basis activities valuations?

Two nice valuations

Theorem

The function

$$F(M) = \sum_{A \subseteq [n]} (A, r_M(A)),$$

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The proofs are based on expressing F and G as linear combinations of indicator functions of suitable polytopes.

A subdivision of $U_{3,6}$

Example

B	M		M_1		M_2		M_3		$M_{1,2}$		$M_{1,3}$		$M_{2,3}$		$M_{1,2,3}$	
	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$
123	$\emptyset$	123	$\emptyset$	123												
124	$\emptyset$	12	$\emptyset$	12												
125	$\emptyset$	12	$\emptyset$	12			$\emptyset$	125			$\emptyset$	125				
126	$\emptyset$	12	5	12			$\emptyset$	12			5	12				
134	$\emptyset$	1	$\emptyset$	1	$\emptyset$	134			$\emptyset$	134						
135	$\emptyset$	1	$\emptyset$	1	$\emptyset$	13	$\emptyset$	15	$\emptyset$	13	$\emptyset$	15	$\emptyset$	135	$\emptyset$	135
136	$\emptyset$	1	5	1	$\emptyset$	13	$\emptyset$	1	5	13	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	15	3	1	3	15	3	15	3	15
146	$\emptyset$	1	35	1	$\emptyset$	1	3	1	35	1	35	1	3	1	35	1
156	$\emptyset$	1					$\emptyset$	1								
234	1	$\emptyset$	1	$\emptyset$	1	34			1	34						
235	1	$\emptyset$	1	$\emptyset$	1	3	1	5	1	3	1	5	1	35	1	35
236	1	$\emptyset$	15	$\emptyset$	1	3	1	$\emptyset$	15	3	15	$\emptyset$	1	3	15	3
245	1	$\emptyset$	13	$\emptyset$	1	$\emptyset$	13	5	13	$\emptyset$	13	5	13	5	13	5
246	1	$\emptyset$	135	$\emptyset$	1	$\emptyset$	13	$\emptyset$	135	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$
256	1	$\emptyset$					1	$\emptyset$								
345	12	$\emptyset$			12	$\emptyset$										
346	12	$\emptyset$			12	$\emptyset$										
356	12	$\emptyset$			12	3	12	$\emptyset$					12	3		
456	123	$\emptyset$			123	$\emptyset$	123	$\emptyset$					123	$\emptyset$		

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126	$\emptyset$	12	5	12			$\emptyset$	12			5	12				
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136	$\emptyset$	1	5	1	$\emptyset$	13	$\emptyset$	1	5	13	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	15	3	1	3	15	3	15	3	15
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123	∅	123	∅	123												
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245	1	∅	13	∅	1	∅	13	5	13	∅	13	5	13	5	13	5
246	1	∅	135	∅	1	∅	13	∅	135	∅	135	∅	13	∅	135	∅
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134	$\emptyset$	1	$\emptyset$	1	$\emptyset$	134			$\emptyset$	134						
135	$\emptyset$	1	$\emptyset$	1	$\emptyset$	13	$\emptyset$	15	$\emptyset$	13	$\emptyset$	15	$\emptyset$	135	$\emptyset$	135
136	$\emptyset$	1	5	1	$\emptyset$	13	$\emptyset$	1	5	13	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	15	3	1	3	15	3	15	3	15
146	$\emptyset$	1	35	1	$\emptyset$	1	3	1	35	1	35	1	3	1	35	1
156	$\emptyset$	1					$\emptyset$	1								
234	1	$\emptyset$	1	$\emptyset$	1	34			1	34						
235	1	$\emptyset$	1	$\emptyset$	1	3	1	5	1	3	1	5	1	35	1	35
236	1	$\emptyset$	15	$\emptyset$	1	3	1	$\emptyset$	15	3	15	$\emptyset$	1	3	15	3
245	1	$\emptyset$	13	$\emptyset$	1	$\emptyset$	13	5	13	$\emptyset$	13	5	13	5	13	5
246	1	$\emptyset$	135	$\emptyset$	1	$\emptyset$	13	$\emptyset$	135	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$
256	1	$\emptyset$					1	$\emptyset$								
345	12	$\emptyset$			12	$\emptyset$										
346	12	$\emptyset$			12	$\emptyset$										
356	12	$\emptyset$			12	3	12	$\emptyset$					12	3		
456	123	$\emptyset$			123	$\emptyset$	123	$\emptyset$					123	$\emptyset$		

A subdivision of $U_{3,6}$

Example

	M		M_1		M_2		M_3		$M_{1,2}$		$M_{1,3}$		$M_{2,3}$		$M_{1,2,3}$	
B	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$
123	$\emptyset$	123	$\emptyset$	123												
124	$\emptyset$	12	$\emptyset$	12												
125	$\emptyset$	12	$\emptyset$	12			$\emptyset$	125			$\emptyset$	125				
126	$\emptyset$	12	5	12			$\emptyset$	12			5	12				
134	$\emptyset$	1	$\emptyset$	1	$\emptyset$	134			$\emptyset$	134						
135	$\emptyset$	1	$\emptyset$	1	$\emptyset$	13	$\emptyset$	15	$\emptyset$	13	$\emptyset$	15	$\emptyset$	135	$\emptyset$	135
136	$\emptyset$	1	5	1	$\emptyset$	13	$\emptyset$	1	5	13	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	15	3	1	3	15	3	15	3	15
146	$\emptyset$	1	35	1	$\emptyset$	1	3	1	35	1	35	1	3	1	35	1
156	$\emptyset$	1					$\emptyset$	1								
234	1	$\emptyset$	1	$\emptyset$	1	34			1	34						
235	1	$\emptyset$	1	$\emptyset$	1	3	1	5	1	3	1	5	1	35	1	35
236	1	$\emptyset$	15	$\emptyset$	1	3	1	$\emptyset$	15	3	15	$\emptyset$	1	3	15	3
245	1	$\emptyset$	13	$\emptyset$	1	$\emptyset$	13	5	13	$\emptyset$	13	5	13	5	13	5
246	1	$\emptyset$	135	$\emptyset$	1	$\emptyset$	13	$\emptyset$	135	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$
256	1	$\emptyset$					1	$\emptyset$								
345	12	$\emptyset$			12	$\emptyset$										
346	12	$\emptyset$			12	$\emptyset$										
356	12	$\emptyset$			12	3	12	$\emptyset$					12	3		
456	123	$\emptyset$			123	$\emptyset$	123	$\emptyset$					123	$\emptyset$		

Some final comments

Using the same techniques one can prove

Theorem

The function

$$H(M) = \sum_{\mathbf{A}} \big((A_1, r_M(A_1)), \ldots, (A_n, r_M(A_n))\big),$$

where $\mathbf{A} = (A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_n)$ *ranges over all maximal flags of* M*, is a valuation.*

This valuation turns out to be *universal* (next talk!).

Some final comments

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This valuation turns out to be *universal* (next talk!).

Future work

Study subdivisions of Coxeter matroids. They appear naturally in similar contexts, like the study of *tropical isotropic linear spaces* (in the case of D_n). What are some nice valuations in this case?

Thank you!