Isotropical Linear Spaces and Valuated Δ-matroids

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Linear subspaces and Grassmannians

Let $K = \bar{K}$, and $m \leq n$. The set of m-dimensional linear subspaces of K^n is parametrized by the Grassmannian $\mathrm{Gr}_{m,n} \subseteq \mathbb{A}^{\binom{[n]}{m}}$ variety:

$$\begin{array}{ccc} \{m\text{-dimensional subspaces}\} & \longrightarrow & \mathrm{Gr}_{m,n} \\ \text{rowspace } A & \longmapsto & (\det A_I)_{I \in \binom{[n]}{m}}. \end{array}$$

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As a subvariety of $\mathbb{A}^{\binom{[n]}{m}}$, $\mathrm{Gr}_{m,n}$ is defined by the ideal generated by the Plücker relations. The shortest ones are called **3-term Plücker relations**:

$$P_{Sab} \cdot P_{Scd} - P_{Sac} \cdot P_{Sbd} + P_{Sad} \cdot P_{Sbc} = 0,$$

where $S \subseteq [n]$ has size $m - 2$, and $a, b, c, d \in [n] - S$ are distinct.

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The support of a Plücker vector $P \in \mathrm{Gr}_{m,n}$

$$\mathrm{supp}\, P := \left\{ I \in \binom{[n]}{m} \;\middle|\; P_I \neq 0 \right\}$$

is the collection of bases of a **matroid** over the ground set $[n] := \{1, 2, \ldots, n\}$.

Tropical Grassmannians

Suppose K is a valuated field. Given a polynomial $f \in K[X_1, \ldots, X_N]$, we can tropicalize it to the "**tropical polynomial**" trop(f) by substituting

$$+ \mapsto \oplus := \min \qquad \cdot \mapsto \odot := + \qquad k \in K \mapsto \mathrm{val}(k) \in \mathbb{T} := \mathbb{Q} \cup \{\infty\}.$$

The tropical hypersurface defined by trop(f) is

$$\mathcal{T}(f) := \{\mathbf{x} \in \mathbb{T}^N \mid \text{the minimum } \mathrm{trop}(f) \text{ is attained at least twice}\}.$$

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Tropical linear spaces

Theorem (Speyer 04)

There exists a correspondence L that makes the following diagram commute:

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In fact, L can be defined for any $p \in \mathrm{Dr}_{m,n}$*:*

- *For any* $T \subseteq [n]$ *of size* $m+1$*, define the "valuated circuit"* $c_T \in \mathbb{T}^n$ *of* p *as* $(c_T)_i := P_{T-i}$.

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$$L(p) := \{\boldsymbol{x} \in \mathbb{T}^n \mid c_1 \odot x_1 \oplus \cdots \oplus c_n \odot x_n \text{ is attained twice for any circuit } c \text{ of } p\}.$$

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Tropical linear spaces satisfy beautiful combinatorics:

- They are pure polyhedral complexes of dimension m.
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Let $p \in \mathbb{T}^{\binom{[n]}{m}}$. Then p is a tropical Plücker vector if and only if the regular subdivision induced by p on the polytope

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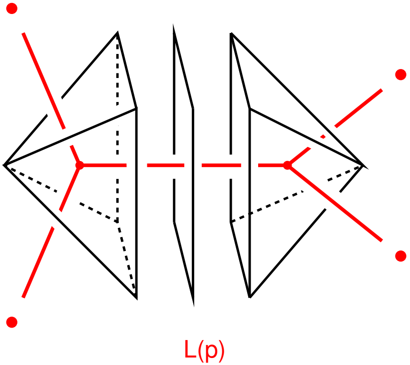
is a matroid subdivision, i.e., all its edges have the form $e_i - e_j$.



Isotropic subspaces and spinor spaces

Denote $\mathbf{2n} := \{1, \ldots, n, 1^*, \ldots, n^*\}$, and let Q be the symmetric bilinear form in $K^{\mathbf{2n}}$

$$Q(x, y) := \sum_{i=1}^{n} x_i \cdot y_{i^*} + \sum_{i=1}^{n} x_{i^*} \cdot y_i.$$

An n-dimensional subspace $U \subseteq K^{\mathbf{2n}}$ is isotropic if $Q(x, y) = 0$ for any $x, y \in U$.

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The set of n-dimensional isotropic subspaces of $K^{\mathbf{2n}}$ is parametrized by the (pure) spinor space $\mathrm{Spin}_n \subseteq \mathbb{A}^{2^{[n]}}$:

$$\begin{array}{ccc} \{n\text{-dimensional isotropic subspaces}\} & \longrightarrow & \mathrm{Spin}_n \\ \text{rowspace}\ (\, I_n \,|\, A_{n\times n} \,) & \longmapsto & (\text{Pfaffian } A_I)_{I \in 2^{[n]}} \end{array}$$

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As a subvariety of $\mathbb{A}^{2^{[n]}}$, Spin_n is defined by the ideal generated by the Wick relations. The shortest ones are called **4-term Wick relations**:

$$W_{Sabcd} \cdot W_S - W_{Sab} \cdot W_{Scd} + W_{Sac} \cdot W_{Sbd} - W_{Sad} \cdot W_{Sbc} = 0$$
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where $S \subseteq [n]$, and $a, b, c, d \in [n] - S$ are distinct.

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The support of a Wick vector $W \in \text{Spin}_n$

$$\text{supp}\, W := \left\{ I \in 2^{[n]} \;\middle|\; W_I \neq 0 \right\}$$

is the collection of bases of an **(even) Δ-matroid** over the ground set $[n]$.

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Definition

A collection of subsets $\mathcal{B} \subseteq 2^{[n]}$ is the set of bases of an (even) Δ-matroid if

- For any $A, B \in \mathcal{B}$ and $i \in A \Delta B$, there exists $j \in B \Delta A$ such that $i \neq j$ and $A \Delta \{i, j\} \in \mathcal{B}$.

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Given any subspace $W \subseteq K^n$, the subspace $U := W \times W^\perp \subseteq K^{\mathbf{2n}}$ is isotropic, so $\text{Gr}_{m,n} \hookrightarrow \text{Spin}_n$.

Moreover, the Plücker vector of W is the same as the Wick vector of U, so Wick vectors **generalize** Plücker vectors, and in particular, Δ-matroids generalize matroids.

Tropical spinor spaces

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The tropical spinor space $\mathrm{TSpin}_n \subseteq \mathbb{T}^{2^{[n]}}$ and the Δ-Dressian $\Delta\mathrm{Dr}_n \subseteq \mathbb{T}^{2^{[n]}}$ are

$$\mathrm{TSpin}_n := \bigcap_{\substack{f \in \text{Wick} \\ \text{ideal}}} \mathcal{T}(f) \qquad \subseteq \qquad \Delta\mathrm{Dr}_n := \bigcap_{\substack{f \text{ Wick} \\ \text{relation}}} \mathcal{T}(f).$$

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Note that tropical Wick vectors **generalize** tropical Plücker vectors.

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Theorem (Jensen-R.)

$\mathrm{TSpin}_n = \Delta\mathrm{Dr}_n$ *if and only if* $n < 6$.

Isotropical linear spaces

Theorem (R.)

*There exists **no** correspondence Q that makes the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Spin}_n & \longleftrightarrow & \{n\text{-dimensional isotropic subspaces}\} \\ \downarrow \text{val} & & \downarrow \text{val} \\ \mathrm{TSpin}_n & \overset{Q}{\longleftrightarrow} & \{n\text{-dimensional isotropical linear spaces}\} \end{array}$$

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In fact, Q can be defined for any $w \in \Delta\text{Dr}_n$:

- *For any* $T \subseteq [n]$, *define a "valuated Δ-circuit"* $c_T \in \mathbb{T}^{2n}$ *of* w.

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$$Q(w) := \{\boldsymbol{x} \in \mathbb{T}^{2n} \cap \mathcal{S} \mid c_1 \odot x_1 \oplus \cdots \oplus c_n \odot x_n \oplus c_{1^*} \odot x_{1^*} \oplus \cdots \oplus c_{n^*} \odot x_{n^*} \text{ is attained twice for any } \Delta\text{-circuit } c \text{ of } w\}.$$

The space $Q(w)$ is called the cocycle space of w, and can be thought of as a generalization of a tropical linear space.

Δ-matroid subdivisions

Cocycle spaces satisfy beautiful combinatorics.

Theorem (Murota 97, R.)

Let $w \in \mathbb{T}^{2^{[n]}}$. Then w is a tropical Wick vector if and only if the regular subdivision induced by w on the polytope

$$\Gamma_w := \text{convex}\{e_I \mid I \in \text{supp}\, w\}$$

is a Δ-matroid subdivision, i.e., all its edges have the form $e_i \pm e_j$.

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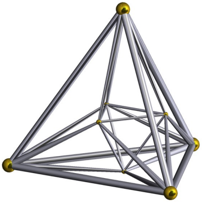
Satisfying the 4-term Wick relations corresponds to subdividing the 4-demicube in a Δ-matroidal way.



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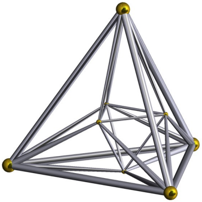

A list of tropical Wick vector spaces and some of their corresponding Δ-matroid subdivisions when $n < 6$ is available at:

`http://math.berkeley.edu/~felipe/delta/`

Cocycle spaces

Theorem (R.)

The cocycle space $Q(w)$ can be described as the intersection of the tropical polytope whose vertices are the cocircuits of w with $\mathcal{S}$, i.e.,

$$Q(w) = \left\{ \bigoplus_{\substack{c \text{ cocircuit} \\ \text{of } w}} \lambda_c \odot c \;\middle|\; \lambda_c \in \mathbb{T} \right\} \cap \mathcal{S}.$$

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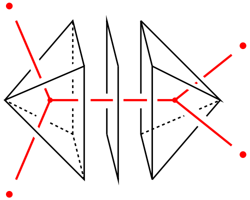
Corollary (Murota-Tamura 01)

*Any tropical linear space $L(p)$ can be described as the **tropical polytope** whose vertices are the cocircuits of p.*



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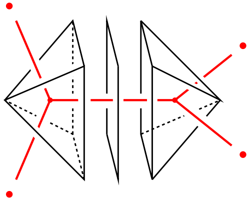
*Any tropical linear space $L(p)$ can be described as the **tropical polytope** whose vertices are the cocircuits of p.*

Corollary (Ardila-Klivans 03)

*If p has only entries in $\{0, \infty\}$ then $L(p)$ can be described as a realization of the **order complex of the lattice of flats** of the matroid associated to p.*



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Thank you!

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