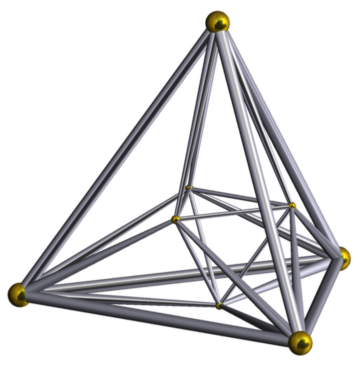

Isotropical Linear Spaces

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Abstract

The spinor variety is defined by the quadratic Wick relations, and its points correspond to maximal isotropic subspaces of a vector space. We tropicalize this picture, and we develop a combinatorial theory of tropical Wick vectors and isotropical linear spaces. We characterize tropical Wick vectors in terms of subdivisions of Δ-matroid polytopes, and we examine to what extent the Wick relations form a tropical basis. Our theory generalizes several results for tropical linear spaces to the class of Coxeter matroids of type D.

1. Grassmannians and Linear Subspaces

Let $K = \bar{K}$, and $m \leq n$. The set of m-dimensional linear subspaces of K^n is parametrized by the **Grassmannian** $\mathrm{Gr}_{m,n} \subseteq \mathbb{A}^{\binom{[n]}{m}}$ variety:

$$\begin{array}{ccc} \{m\text{-dimensional subspaces}\} & \longrightarrow & \mathrm{Gr}_{m,n} \\ \text{rowspace } A & \longmapsto & (\det A_I)_{I \in \binom{[n]}{m}}. \end{array}$$

As a subvariety of $\mathbb{A}^{\binom{[n]}{m}}$, $\mathrm{Gr}_{m,n}$ is defined by the ideal generated by the **Plücker relations**. The shortest ones are called **3-term Plücker relations**:

$$P_{Sab} \cdot P_{Scd} - P_{Sac} \cdot P_{Sbd} + P_{Sad} \cdot P_{Sbc} = 0,$$

where $S \subseteq [n]$ has size $m-2$, and $a, b, c, d \in [n] - S$ are distinct.

2. Tropical Grassmannians and Linear Spaces

Let $\mathbb{T} = \mathbb{R} \cup \{\infty\}$. The **tropical Grassmannian** $\mathrm{TGr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$ and the **Dressian** $\mathrm{Dr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$ are defined as

$$\mathrm{TGr}_{m,n} := \bigcap_{\substack{f \in \text{Plücker} \\ \text{ideal}}} \mathcal{T}(f) \quad \subseteq \quad \mathrm{Dr}_{m,n} := \bigcap_{\substack{f \text{ Plücker} \\ \text{relation}}} \mathcal{T}(f).$$

Theorem ([Spe08]). *There exists a correspondence L that makes the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Gr}_{m,n} & \longleftrightarrow & \{m\text{-dimensional subspaces of } K^n\} \\ \downarrow \text{val} & & \downarrow \text{val} \\ \mathrm{TGr}_{m,n} & \overset{L}{\longleftrightarrow} & \{m\text{-dimensional } \textbf{tropicalized} \text{ linear spaces}\} \end{array}$$

In fact, L can be defined for any $p \in \mathrm{Dr}_{m,n}$ as

$L(p) := \{x \in \mathbb{T}^n \mid c_1 \odot x_1 \oplus \cdots \oplus c_n \odot x_n$ *is attained twice for any* ***circuit*** *c of p*$\}$.

3. Isotropic Subspaces and Spinors

Denote $\mathbf{2n} := \{1, \ldots, n, 1^*, \ldots, n^*\}$, and let Q be the symmetric bilinear form in $K^{\mathbf{2n}}$

$$Q(x,y) := \sum_{i=1}^{n} x_i \cdot y_{i^*} + \sum_{i=1}^{n} x_{i^*} \cdot y_i.$$

An n-dimensional subspace $U \subseteq K^{\mathbf{2n}}$ is **isotropic** if $Q(x,y) = 0$ for any $x, y \in U$.

The set of n-dimensional isotropic subspaces of $K^{\mathbf{2n}}$ is parametrized by the **(pure) spinor space** $\mathrm{Spin}_n \subseteq \mathbb{A}^{2^{[n]}}$:

$$\begin{array}{ccc} \{n\text{-dimensional isotropic subspaces}\} & \longrightarrow & \mathrm{Spin}_n \\ \text{rowspace } (\, I_n \mid A_{n \times n} \,) & \longmapsto & (\text{Pfaffian } A_I)_{I \in 2^{[n]}} \end{array}$$

As a subvariety of $\mathbb{A}^{2^{[n]}}$, Spin_n is defined by the ideal generated by the **Wick relations**. The shortest ones are called **4-term Wick relations**:

$$W_{Sabcd} \cdot W_S - W_{Sab} \cdot W_{Scd} + W_{Sac} \cdot W_{Sbd} - W_{Sad} \cdot W_{Sbc} = 0$$
$$W_{Sabc} \cdot W_{Sd} - W_{Sabd} \cdot W_{Sc} + W_{Sacd} \cdot W_{Sb} - W_{Sbcd} \cdot W_{Sa} = 0$$

where $S \subseteq [n]$, and $a, b, c, d \in [n] - S$ are distinct.

Given any subspace $W \subseteq K^n$, the subspace $U := W \times W^{\perp} \subseteq K^{\mathbf{2n}}$ is isotropic, so $\mathrm{Gr}_{m,n} \hookrightarrow \mathrm{Spin}_n$. Moreover, the Plücker vector of W is the same as the Wick vector of U, so Wick vectors **generalize** Plücker vectors.

4. Isotropical Linear Spaces

The **tropical spinor space** $\mathrm{TSpin}_n \subseteq \mathbb{T}^{2^{[n]}}$ and the **Δ-Dressian** $\Delta\mathrm{Dr}_n \subseteq \mathbb{T}^{2^{[n]}}$ are defined as

$$\mathrm{TSpin}_n := \bigcap_{\substack{f \in \text{Wick} \\ \text{ideal}}} \mathcal{T}(f) \quad \subseteq \quad \Delta\mathrm{Dr}_n := \bigcap_{\substack{f \text{ Wick} \\ \text{relation}}} \mathcal{T}(f).$$

Theorem (R.). *There exists a correspondence Q that makes the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Spin}_n & \longleftrightarrow & \{n\text{-dimensional isotropic subspaces}\} \\ \downarrow \text{val} & & \downarrow \text{val} \\ \mathrm{TSpin}_n & \overset{Q}{\longleftrightarrow} & \{n\text{-dimensional tropicalized } \textbf{isotropic} \text{ linear spaces} \cap \mathcal{S}\} \end{array}$$

where $\mathcal{S} := \{x \in \mathbb{T}^{\mathbf{2n}} \mid$ for any $i \in [n]$, at least one of x_i and x_{i^*} is equal to $\infty\}$.

In fact, Q can be defined for any $w \in \Delta\mathrm{Dr}_n$ as

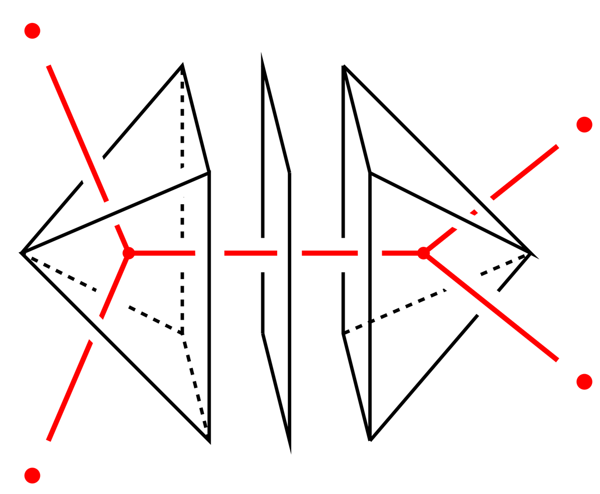
$Q(w) := \{x \in \mathbb{T}^{\mathbf{2n}} \cap \mathcal{S} \mid c_1 \odot x_1 \oplus \cdots \oplus c_n \odot x_n \oplus c_{1^*} \odot x_{1^*} \oplus \cdots \oplus c_{n^*} \odot x_{n^*}$ *is attained twice for any* ***circuit*** *c of w*$\}$.

5. Wick Relations and Tropical Bases

Theorem (Jensen-R.). $\mathrm{TSpin}_n = \Delta\mathrm{Dr}_n$ *if and only if* $n < 6$.

6. Matroid Polytope Subdivisions

Theorem ([Spe08]). *Let $p \in \mathbb{T}^{\binom{[n]}{m}}$. Then $p \in \mathrm{Dr}_{m,n}$ if and only if the regular subdivision induced by p on the polytope $\Gamma_p := \mathrm{convex}\{e_I \mid p_I \neq \infty\}$ is a* ***matroid subdivision****, i.e., all its edges have the form $e_i - e_j$.*



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7. Δ-matroid Polytope Subdivisions

Theorem (R.). *Let $w \in \mathbb{T}^{2^{[n]}}$. Then w is a tropical Wick vector if and only if the regular subdivision induced by w on the polytope $\Gamma_w := \mathrm{convex}\{e_I \mid w_I \neq \infty\}$ is a* ***Δ-matroid subdivision****, i.e., all its edges have the form $e_i \pm e_j$.*

These Δ-matroids can be thought of as **Coxeter matroids** of type D [BGW03].

8. Tropical Linear Spaces and Tropical Polytopes

Theorem (R.). *The space $Q(w)$ can be described as the intersection of the* ***tropical polytope*** *whose vertices are the* ***cocircuits*** *of w with $\mathcal{S}$, i.e.,*

$$Q(w) = \Big\{ \bigoplus_{\substack{c \text{ cocircuit} \\ \text{of } w}} \lambda_c \odot c \;\Big|\; \lambda_c \in \mathbb{T} \Big\} \cap \mathcal{S}.$$

Corollary ([MT01]). *Any tropical linear space $L(p)$ can be described as the* ***tropical polytope*** *whose vertices are the* ***cocircuits*** *of p.*

Corollary ([AK06]). *If p has only entries in $\{0, \infty\}$ then $L(p)$ can be described as a realization of the* ***order complex of the lattice of flats*** *of the matroid associated to p.*

References

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