Computing tropical linear spaces and A-discriminants

Felipe Rincón

Department of Mathematics
University of California, Berkeley

Outline

What is tropical geometry?

Tropical geometry is a tool for transforming classical algebraic varieties into polyhedral objects called tropical varieties. These “simplifications” retain some information about the original variety.

What is tropical geometry?

Tropical geometry is a tool for transforming classical algebraic varieties into polyhedral objects called tropical varieties. These "simplifications" retain some information about the original variety.

Definition

The field $\mathbb{C}\{\{t\}\}$ of **Puiseux series** on the variable t is the set of formal series of the form

$$f = \sum_{k=k_0}^{+\infty} c_k \cdot t^{\frac{k}{N}},$$

where $N \in \mathbb{Z}_{>0}$, $k_0 \in \mathbb{Z}$, and all $c_k \in \mathbb{C}$.

What is tropical geometry?

Tropical geometry is a tool for transforming classical algebraic varieties into polyhedral objects called tropical varieties. These "simplifications" retain some information about the original variety.

Definition

The field $\mathbb{C}\{\{t\}\}$ of **Puiseux series** on the variable t is the set of formal series of the form

$$f = \sum_{k=k_0}^{+\infty} c_k \cdot t^{\frac{k}{N}},$$

where $N \in \mathbb{Z}_{>0}$, $k_0 \in \mathbb{Z}$, and all $c_k \in \mathbb{C}$.

The field of Puiseux series is algebraically closed, and it comes equipped with a **valuation** map $\text{val} : \mathbb{C}\{\{t\}\} - 0 \to \mathbb{Q}$ defined as

$$\text{val}(f) = \min \left\{ \frac{k}{N} \;\middle|\; c_k \neq 0 \right\}.$$

What is tropical geometry?

Tropical geometry is a tool for transforming classical algebraic varieties into polyhedral objects called tropical varieties. These "simplifications" retain some information about the original variety.

Definition

The field $\mathbb{C}\{\{t\}\}$ of **Puiseux series** on the variable t is the set of formal series of the form

$$f = \sum_{k=k_0}^{+\infty} c_k \cdot t^{\frac{k}{N}},$$

where $N \in \mathbb{Z}_{>0}$, $k_0 \in \mathbb{Z}$, and all $c_k \in \mathbb{C}$.

The field of Puiseux series is algebraically closed, and it comes equipped with a **valuation** map $\text{val} : \mathbb{C}\{\{t\}\} - 0 \to \mathbb{Q}$ defined as

$$\text{val}(f) = \min \left\{ \frac{k}{N} \;\middle|\; c_k \neq 0 \right\}.$$

Example

$$f = 2 \cdot t^{-5/3} - 7 \cdot t^{-1} - 3i \cdot t^{-2/3} + t^{1/3} + \cdots, \qquad \text{val}(f) = -5/3.$$

Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The **tropicalization** of V is the set

$$\mathcal{T}(V) := \{(\mathrm{val}(f_1), \ldots, \mathrm{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The **tropicalization** of V is the set

$$\mathcal{T}(V) := \{(\mathrm{val}(f_1), \ldots, \mathrm{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

Example

Take $I = \langle X + Y + Z \rangle$. The variety V is a plane in $(\mathbb{C}\{\{t\}\} - 0)^3$.

Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The **tropicalization** of V is the set

$$\mathcal{T}(V) := \{(\mathrm{val}(f_1), \ldots, \mathrm{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

Example

Take $I = \langle X + Y + Z \rangle$. The variety V is a plane in $(\mathbb{C}\{\{t\}\} - 0)^3$. A point $(f_1, f_2, f_3) \in V$ has to satisfy one of the following:

$$\mathrm{val}(f_1) = \mathrm{val}(f_2) \leq \mathrm{val}(f_3),$$
$$\mathrm{val}(f_1) = \mathrm{val}(f_3) \leq \mathrm{val}(f_2),$$
$$\mathrm{val}(f_2) = \mathrm{val}(f_3) \leq \mathrm{val}(f_1);$$

so $\min(x, y, z)$ is attained twice.

Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The **tropicalization** of V is the set

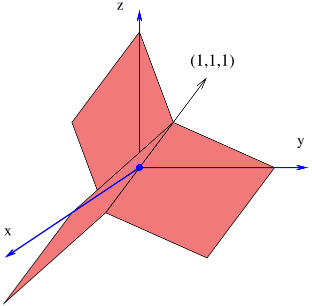
$$\mathcal{T}(V) := \{(\mathrm{val}(f_1), \ldots, \mathrm{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

Example

Take $I = \langle X + Y + Z \rangle$. The variety V is a plane in $(\mathbb{C}\{\{t\}\} - 0)^3$. A point $(f_1, f_2, f_3) \in V$ has to satisfy one of the following:

$$\mathrm{val}(f_1) = \mathrm{val}(f_2) \leq \mathrm{val}(f_3),$$
$$\mathrm{val}(f_1) = \mathrm{val}(f_3) \leq \mathrm{val}(f_2),$$
$$\mathrm{val}(f_2) = \mathrm{val}(f_3) \leq \mathrm{val}(f_1);$$

so $\min(x, y, z)$ is attained twice.



Tropicalizing varieties

Suppose $I \subseteq \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is an ideal and $V \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$ its corresponding variety. The **tropicalization** of V is the set

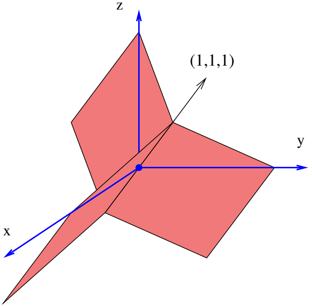
$$\mathcal{T}(V) := \{(\text{val}(f_1), \ldots, \text{val}(f_n)) \mid (f_1, \ldots, f_n) \in V\} \subseteq \mathbb{Q}^n.$$

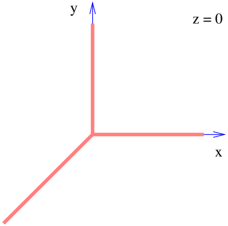
Example

Take $I = \langle X + Y + Z \rangle$. The variety V is a plane in $(\mathbb{C}\{\{t\}\} - 0)^3$. A point $(f_1, f_2, f_3) \in V$ has to satisfy one of the following:

$$\text{val}(f_1) = \text{val}(f_2) \leq \text{val}(f_3),$$
$$\text{val}(f_1) = \text{val}(f_3) \leq \text{val}(f_2),$$
$$\text{val}(f_2) = \text{val}(f_3) \leq \text{val}(f_1);$$

so $\min(x, y, z)$ is attained twice.





Tropical hypersurfaces

In general, suppose

$$p := \sum c_\alpha \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n} \quad \in \quad \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}],$$

and let $V := V(p) \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$. The tropicalization $\mathcal{T}(V)$ is then

$$\mathcal{T}(V) = \left\{ w \in \mathbb{Q}^n \;\middle|\; \min\{w_1 \cdot \alpha_1 + \cdots + w_n \cdot \alpha_n \mid c_\alpha \neq 0\} \text{ is attained twice} \right\}.$$

Tropical hypersurfaces

In general, suppose

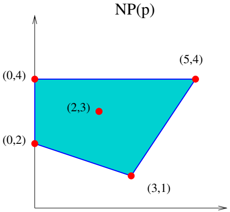
$$p := \sum c_\alpha \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n} \quad \in \quad \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}],$$

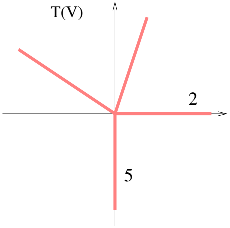
and let $V := V(p) \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$. The tropicalization $\mathcal{T}(V)$ is then

$$\mathcal{T}(V) = \left\{ w \in \mathbb{Q}^n \;\middle|\; \min\{w_1 \cdot \alpha_1 + \cdots + w_n \cdot \alpha_n \mid c_\alpha \neq 0\} \text{ is attained twice} \right\}.$$

Example

Take $p = X^5Y^4 + 2X^2Y^3 - 7X^3Y + 3Y^4 - Y^2$. The tropicalization of $V := V(p)$ is the (negative of the) one-dimensional skeleton of the normal fan of the **Newton polytope** of p.





Tropical hypersurfaces

In general, suppose

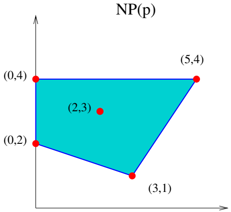
$$p := \sum c_\alpha \cdot X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}],$$

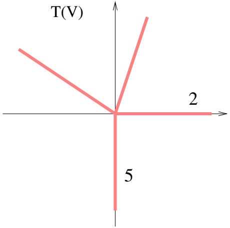
and let $V := V(p) \subseteq (\mathbb{C}\{\{t\}\} - 0)^n$. The tropicalization $\mathcal{T}(V)$ is then

$$\mathcal{T}(V) = \left\{ w \in \mathbb{Q}^n \;\middle|\; \min\{w_1 \cdot \alpha_1 + \cdots + w_n \cdot \alpha_n \mid c_\alpha \neq 0\} \text{ is attained twice} \right\}.$$

Example

Take $p = X^5Y^4 + 2X^2Y^3 - 7X^3Y + 3Y^4 - Y^2$. The tropicalization of $V := V(p)$ is the (negative of the) one-dimensional skeleton of the normal fan of the **Newton polytope** of p.





Tropical geometry is concerned with the study of these **tropical varieties**. Simply put, tropical geometry is the geometry of the leading exponents.

Tropical linear spaces

Let L be the rowspan over $\mathbb{C}\{\{t\}\}$ of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Tropical linear spaces

Let L be the rowspan over $\mathbb{C}\{\{t\}\}$ of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

The kernel of A is the rowspan of

$$A^{\perp} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix},$$

so $I(L) = \langle 2X_1 + X_2 - X_3, 4X_1 + 2X_2 - X_4 \rangle$.

Tropical linear spaces

Let L be the rowspan over $\mathbb{C}\{\{t\}\}$ of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

The kernel of A is the rowspan of

$$A^{\perp} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix},$$

so $I(L) = \langle 2X_1 + X_2 - X_3, 4X_1 + 2X_2 - X_4 \rangle$.

The tropicalization $\mathcal{T}(L)$ is then in the intersection of the fans

$\min(x_1, x_2, x_3)$ is attained twice, $\min(x_1, x_2, x_4)$ is attained twice.

Tropical linear spaces

Let L be the rowspan over $\mathbb{C}\{\{t\}\}$ of the matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & 6 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

The kernel of A is the rowspan of

$$A^{\perp} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix},$$

so $I(L) = \langle 2X_1 + X_2 - X_3, 4X_1 + 2X_2 - X_4 \rangle$.

The tropicalization $\mathcal{T}(L)$ is then in the intersection of the fans

$\min(x_1, x_2, x_3)$ is attained twice, $\min(x_1, x_2, x_4)$ is attained twice.

However, note that there is one more relation among the columns of A, namely $2X_3 - X_4 \in I(L)$. So we need one more condition:

$\min(x_3, x_4)$ is attained twice.

Tropical linear spaces

In general, suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$C \subseteq \{1, \ldots, n\}$ is a **circuit** of $A \iff \{\mathbf{a}_i \mid i \in C\}$ is minimally dependent.

Tropical linear spaces

In general, suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$$C \subseteq \{1, \ldots, n\} \text{ is a } \textbf{circuit} \text{ of } A \iff \{\mathbf{a}_i \mid i \in C\} \text{ is minimally dependent.}$$

Theorem

*The tropicalization $\mathcal{T}(L)$ depends only on the set of circuits of A (its **matroid**). In fact,*

$$w \in \mathcal{T}(L) \iff \textit{for any circuit } C,\ \min(w_i \mid i \in C) \textit{ is attained twice.}$$

Tropical linear spaces

In general, suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$$C \subseteq \{1, \ldots, n\} \text{ is a } \textbf{circuit} \text{ of } A \iff \{\mathbf{a}_i \mid i \in C\} \text{ is minimally dependent.}$$

Theorem

The tropicalization $\mathcal{T}(L)$ depends only on the set of circuits of A (its ***matroid****). In fact,*

$$w \in \mathcal{T}(L) \iff \text{for any circuit } C,\ \min(w_i \mid i \in C) \text{ is attained twice.}$$

- This is not a computationally nice description of $\mathcal{T}(L)$, as it is given as an intersection of many fans.

Tropical linear spaces

In general, suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$$C \subseteq \{1, \ldots, n\} \text{ is a } \textbf{circuit} \text{ of } A \iff \{\mathbf{a}_i \mid i \in C\} \text{ is minimally dependent.}$$

Theorem

*The tropicalization $\mathcal{T}(L)$ depends only on the set of circuits of A (its **matroid**). In fact,*

$$w \in \mathcal{T}(L) \iff \textit{for any circuit } C, \ \min(w_i \mid i \in C) \textit{ is attained twice.}$$

- This is not a computationally nice description of $\mathcal{T}(L)$, as it is given as an intersection of many fans.
- Ardila and Klivans gave a parametric description of $\mathcal{T}(L)$ in terms of all chains of **flats** of A, but computing (chains of) flats is hard.

Tropical linear spaces

In general, suppose L is the rowspace over $\mathbb{C}\{\{t\}\}$ of an $m \times n$ integer matrix A with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Definition

$$C \subseteq \{1, \ldots, n\} \text{ is a } \textbf{circuit} \text{ of } A \iff \{\mathbf{a}_i \mid i \in C\} \text{ is minimally dependent.}$$

Theorem

*The tropicalization $\mathcal{T}(L)$ depends only on the set of circuits of A (its **matroid**). In fact,*

$$w \in \mathcal{T}(L) \iff \textit{for any circuit } C, \ \min(w_i \mid i \in C) \textit{ is attained twice.}$$

- This is not a computationally nice description of $\mathcal{T}(L)$, as it is given as an intersection of many fans.
- Ardila and Klivans gave a parametric description of $\mathcal{T}(L)$ in terms of all chains of **flats** of A, but computing (chains of) flats is hard.

Why do we care?

A-discriminants

Suppose A is an $m \times n$ integer matrix. The columns of A can be identified with Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\mathbb{C}^A$ be the space of all Laurent polynomials of the form $p(\mathbf{x}) = \sum_{i=1}^{n} c_i \cdot \mathbf{x}^{\mathbf{a}_i}$.

A-discriminants

Suppose A is an $m \times n$ integer matrix. The columns of A can be identified with Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\mathbb{C}^A$ be the space of all Laurent polynomials of the form $p(\mathbf{x}) = \sum_{i=1}^n c_i \cdot \mathbf{x}^{\mathbf{a}_i}$.

The **discriminantal variety** ∇_A is the Zariski closure of the set of polynomials p in $\mathbb{C}^A$ for which there exists a $\mathbf{z} \in (\mathbb{C}^*)^m$ satisfying

$$p(\mathbf{z}) = \frac{\partial p}{\partial x_i}(\mathbf{z}) = 0 \quad \text{for all } i = 1, \ldots, m.$$

A-discriminants

Suppose A is an $m \times n$ integer matrix. The columns of A can be identified with Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\mathbb{C}^A$ be the space of all Laurent polynomials of the form $p(\mathbf{x}) = \sum_{i=1}^n c_i \cdot \mathbf{x}^{\mathbf{a}_i}$.

The **discriminantal variety** ∇_A is the Zariski closure of the set of polynomials p in $\mathbb{C}^A$ for which there exists a $\mathbf{z} \in (\mathbb{C}^*)^m$ satisfying

$$p(\mathbf{z}) = \frac{\partial p}{\partial x_i}(\mathbf{z}) = 0 \quad \text{for all } i = 1, \ldots, m.$$

Example

Take $A = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$, so $\mathbb{C}^A = \{p = a \cdot x^2 + b \cdot x + c \mid a, b, c \in \mathbb{C}\}$. Then $\nabla_A = \{p \in \mathbb{C}^A \mid b^2 - 4ac = 0\}$.

A-discriminants

Suppose A is an $m \times n$ integer matrix. The columns of A can be identified with Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. Let $\mathbb{C}^A$ be the space of all Laurent polynomials of the form $p(\mathbf{x}) = \sum_{i=1}^n c_i \cdot \mathbf{x}^{\mathbf{a}_i}$.

The **discriminantal variety** ∇_A is the Zariski closure of the set of polynomials p in $\mathbb{C}^A$ for which there exists a $\mathbf{z} \in (\mathbb{C}^*)^m$ satisfying

$$p(\mathbf{z}) = \frac{\partial p}{\partial x_i}(\mathbf{z}) = 0 \quad \text{for all } i = 1, \ldots, m.$$

Example

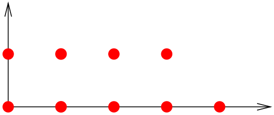
Take $A = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$, so $\mathbb{C}^A = \{p = a \cdot x^2 + b \cdot x + c \mid a, b, c \in \mathbb{C}\}$. Then $\nabla_A = \{p \in \mathbb{C}^A \mid b^2 - 4ac = 0\}$.

If ∇_A has codimension 1 then its defining polynomial Δ_A is called the A-**discriminant**.

An example

Let

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$



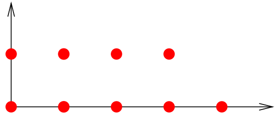
Then $\mathbb{C}^A$ is the space of polynomials

$$p(x,y) = a \cdot x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x + e + f \cdot x^3 y + g \cdot x^2 y + h \cdot xy + i \cdot y.$$

An example

Let

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$



Then $\mathbb{C}^A$ is the space of polynomials

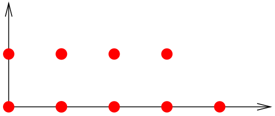
$$p(x,y) = a \cdot x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x + e + f \cdot x^3 y + g \cdot x^2 y + h \cdot xy + i \cdot y.$$

The A-discriminant Δ_A can be computed to be

An example

Let

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$



Then $\mathbb{C}^A$ is the space of polynomials

$$p(x,y) = a \cdot x^4 + b \cdot x^3 + c \cdot x^2 + d \cdot x + e + f \cdot x^3 y + g \cdot x^2 y + h \cdot xy + i \cdot y.$$

The A-discriminant Δ_A can be computed to be

$$e^3f^4 - de^2f^3g + ce^2f^2g^2 - be^2fg^3 + ae^2g^4 + d^2ef^3h - 2ce^2f^3h - cdef^2gh + 3be^2f^2gh + bdefg^2h - 4ae^2fg^2h - adeg^3h + c^2ef^2h^2 - 2bdef^2h^2 + 2ae^2f^2h^2 - bcefgh^2 + 3adefgh^2 + aceg^2h^2 + b^2efh^3 - 2acefh^3 - abegh^3 + a^2eh^4 - d^3f^3i + 3cdef^3i - 3be^2f^3i + cd^2f^2gi - 2c^2ef^2gi - bdef^2gi + 4ae^2f^2gi - bd^2fg^2i + 2bcefg^2i + adefg^2i + ad^2g^3i - 2aceg^3i - c^2df^2hi + 2bd^2f^2hi + bcef^2hi - 5adef^2hi + bcdfghi - 3ad^2fghi - 3b^2efghi + 4acefghi - acdg^2hi + 3abeg^2hi - b^2dfh^2i + 2acdfh^2i + abefh^2i + abdgh^2i - 4a^2egh^2i - a^2dh^3i + c^3f^2i^2 - 3bcdf^2i^2 + 3ad^2f^2i^2 + 3b^2ef^2i^2 - 3acef^2i^2 - bc^2fgi^2 + 2b^2dfgi^2 + acdfgi^2 - 5abefgi^2 + ac^2g^2i^2 - 2abdg^2i^2 + 2a^2eg^2i^2 + b^2cfhi^2 - 2ac^2fhi^2 - abdfhi^2 + 4a^2efhi^2 - abcghi^2 + 3a^2dghi^2 + a^2ch^2i^2 - b^3fi^3 + 3abcfi^3 - 3a^2dfi^3 + ab^2gi^3 - 2a^2cgi^3 - a^2bhi^3 + a^3i^4.$$

One more example

Take

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The A-discriminant Δ_A expresses the condition on the coefficients of two general quadrics for them to be tangent.

One more example

Take

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The A-discriminant Δ_A expresses the condition on the coefficients of two general quadrics for them to be tangent.

It is a surprisingly large homogeneous polynomial of degree 12 having 3210 different monomials!

One more example

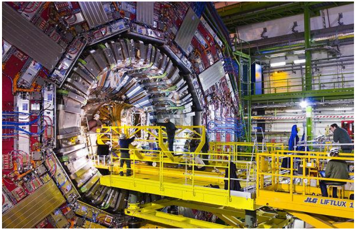
Take

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The A-discriminant Δ_A expresses the condition on the coefficients of two general quadrics for them to be tangent.

It is a surprisingly large homogeneous polynomial of degree 12 having 3210 different monomials!

It was used by physicist C. Lester to "write M_{T2}^2 as the root of a single order 4 polynomial. This will permit us to calculate it at a rate of 40 MHz, which will allow us to trigger the ATLAS detector at the Large Hadron Collider to take pictures of super-symmetric particles (if they exist). This discriminant is instrumental in reaching that 40 MHz bunch crossing rate! :-) ".



One more example

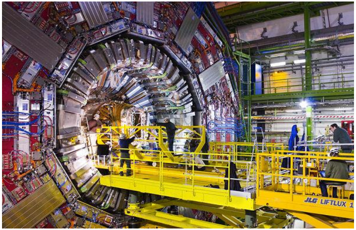
Take

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The A-discriminant Δ_A expresses the condition on the coefficients of two general quadrics for them to be tangent.

It is a surprisingly large homogeneous polynomial of degree 12 having 3210 different monomials!

It was used by physicist C. Lester to "write M_{T2}^2 as the root of a single order 4 polynomial. This will permit us to calculate it at a rate of 40 MHz, which will allow us to trigger the ATLAS detector at the Large Hadron Collider to take pictures of super-symmetric particles (if they exist). This discriminant is instrumental in reaching that 40 MHz bunch crossing rate! :-) ".



So discriminants are interesting and useful, but also large and hard to compute.

The tropical approach

Suppose the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let T be the image of the monomial map

$$\begin{array}{ccc} (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^n \\ \mathbf{t} & \longmapsto & (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}). \end{array}$$

Theorem (Horn uniformization)

The discriminantal variety ∇_A is the Zariski closure of the coordinate-wise product between $\ker(A)$ *and T, i.e.,*

$$\nabla_A = \overline{\ker(A) \cdot T}.$$

The tropical approach

Suppose the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let T be the image of the monomial map

$$\begin{array}{rcl} (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^n \\ \mathbf{t} & \longmapsto & (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}). \end{array}$$

Theorem (Horn uniformization)

The discriminantal variety ∇_A is the Zariski closure of the coordinate-wise product between ker(A) *and T, i.e.,*

$$\nabla_A = \overline{\ker(A) \cdot T}.$$

Corollary (Dickenstein-Feichtner-Sturmfels)

*The **tropical discriminantal variety** can be described as*

$$\mathcal{T}(\nabla_A) = \mathcal{T}(\ker(A)) + \text{rowspace}(A).$$

The tropical approach

Suppose the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Let T be the image of the monomial map

$$\begin{array}{rcl} (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^n \\ \mathbf{t} & \longmapsto & (\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}). \end{array}$$

Theorem (Horn uniformization)

The discriminantal variety ∇_A is the Zariski closure of the coordinate-wise product between ker(A) *and T, i.e.,*

$$\nabla_A = \overline{\ker(A) \cdot T}.$$

Corollary (Dickenstein-Feichtner-Sturmfels)

The ***tropical discriminantal variety*** *can be described as*

$$\mathcal{T}(\nabla_A) = \mathcal{T}(\ker(A)) + \text{rowspace}(A).$$

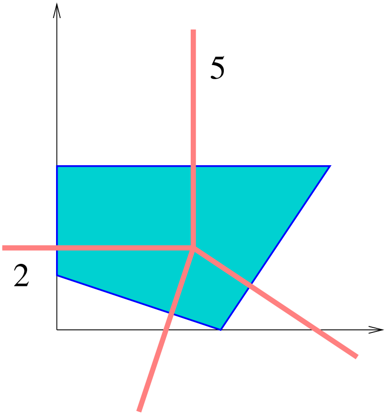
This can be used to compute $\mathcal{T}(\nabla_A)$, if we can effectively compute $\mathcal{T}(\ker(A))$.

How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.

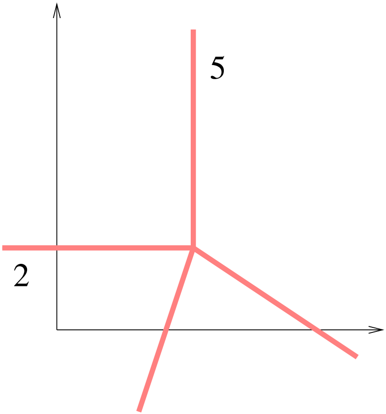
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



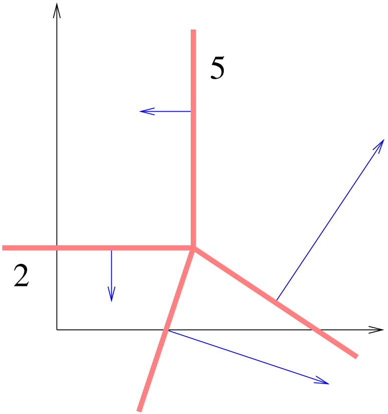
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



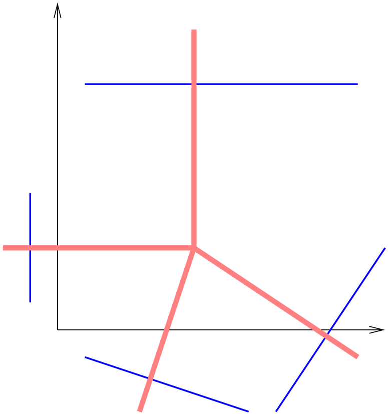
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



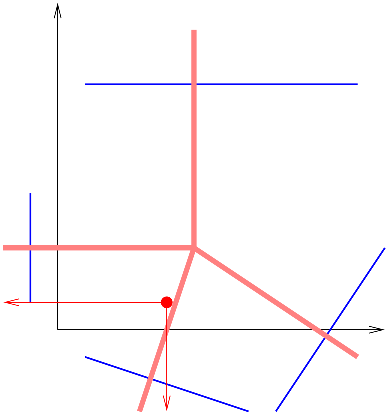
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



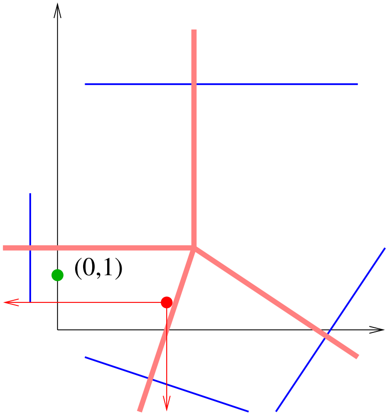
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



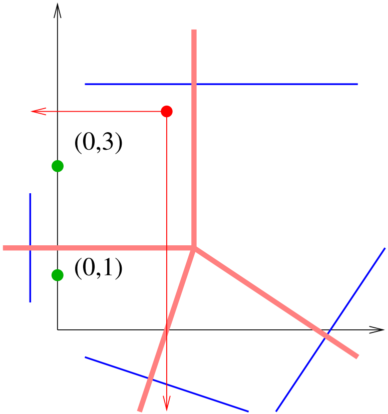
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



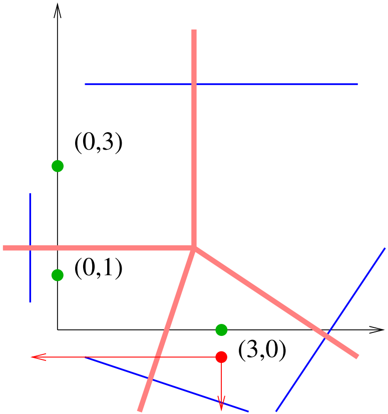
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



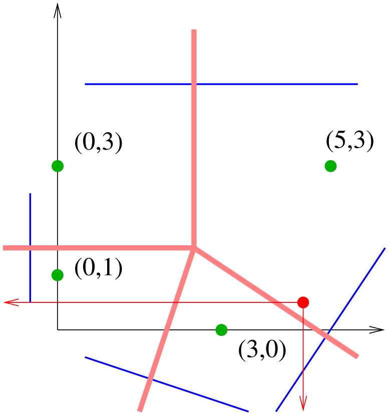
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the *A*-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



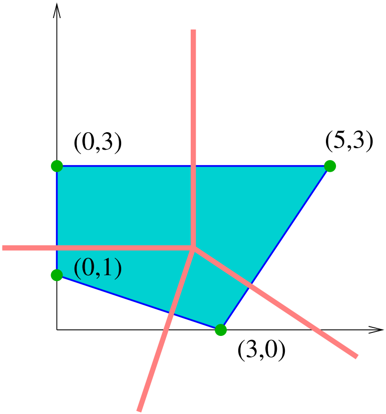
How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



How to recover Δ_A?

If ∇_A has codimension 1, we can recover the Newton polytope of the A-discriminant Δ_A from $\mathcal{T}(\nabla_A)$ by using a **ray shooting algorithm**.



If we can compute tropical linear spaces effectively then we can compute (the Newton polytope) of Δ_A.

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \ldots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \ldots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \ldots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \ldots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \dots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \dots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \dots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \dots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix}$$

$$B = \{1, 3, 5\}$$

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \ldots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \ldots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix}$$

$$w = (2, 1, 1, 3, 3, 1) \in \Sigma_B$$

$$B = \{1, 3, 5\}$$

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \ldots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \ldots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix}$$

$$B = \{1, 3, 5\}$$

$$w = (2, 1, 1, 3, 3, 1) \in \Sigma_B$$

$$C(2, B) = \{1, 2, 3\}, \; C(4, B) = \{4, 5\},$$
$$C(6, B) = \{1, 3, 5, 6\}$$

A local criterion for tropical linear spaces

Suppose A is an $m \times n$ integer matrix of rank m with columns labeled by the set $\{1, \dots, n\}$, and let $L = \text{rowspace}(A)$. Suppose the columns in $B \subseteq \{1, \dots, n\}$ form a **basis** for $\mathbb{R}^m$.

Definition

For any $k \notin B$ there is a unique (minimal) dependence among the columns in $B \cup \{k\}$, called the **fundamental circuit** $C(k, B)$ of k over B.

Definition

Given a vector $w \in \mathbb{Q}^n$, the w-**weight** of a basis B' is $\sum_{i \in B'} w_i$. For any basis B, let Σ_B be the set of vectors w such that B is a basis of maximal w-weight.

Theorem (Feichtner-Sturmfels)

Suppose $w \in \Sigma_B$. Then w is in $\mathcal{T}(L)$ if and only if $\min\{w_i \mid i \in C\}$ is attained twice for any fundamental circuit C over the basis B.

Example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & -1 \end{pmatrix}$$

$$B = \{1, 3, 5\}$$

$$w = (2, 1, 1, 3, 3, 1) \in \Sigma_B$$

$$C(2, B) = \{1, 2, 3\}, \; C(4, B) = \{4, 5\}, \; C(6, B) = \{1, 3, 5, 6\}$$

$$w \in \mathcal{T}(L)$$

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$w = (0, 5, 2, 3, 3, 0, 0)$$

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$w = (0, 5, 2, 3, 3, 0, 0)$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$w = (0, 5, 2, 3, 3, 0, 0)$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

Total order J: $1 <_J 3 <_J 4 <_J 2$

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$w = (0, 5, 2, 3, 3, 0, 0)$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

Total order J: $1 <_J 3 <_J 4 <_J 2$

Preference function p: $5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1$

The cyclic Bergman fan

A generic vector $w \in \Sigma_B \cap \mathcal{T}(L)$ induces a **preference function** $p : B^c \to B$ in the following way:

1. Define the total order J on B as $a <_J b \iff w_a < w_b$.
2. For any $k \notin B$ define $p(k) =$ "J-smallest element of $C(k, B) - \{k\}$".

Example

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$w = (0, 5, 2, 3, 3, 0, 0)$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

Total order J: $1 <_J 3 <_J 4 <_J 2$

Preference function p: $5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1$

We have $\min\{w_i \mid i \in C(k, B)\} = w_k = w_{p(k)}$, so p encodes which coordinates attain the minimum in each fundamental circuit.

The cyclic Bergman fan

Let K be the restriction of the order J to the image of p. We will say that the pair (p, K) is the **compatible pair** induced by w.

The cyclic Bergman fan

Let K be the restriction of the order J to the image of p. We will say that the pair (p, K) is the **compatible pair** induced by w.

Proposition

Let (p, K) be a compatible pair. The set of vectors $w \in \Sigma_B \cap \mathcal{T}(L)$ that induce (p, K) is an m-dimensional polyhedral cone $\Gamma(p, K)$ in $\mathbb{R}^n$ whose extremal rays can all be taken to be $0/1$ vectors.

The cyclic Bergman fan

Let K be the restriction of the order J to the image of p. We will say that the pair (p, K) is the **compatible pair** induced by w.

Proposition

Let (p, K) be a compatible pair. The set of vectors $w \in \Sigma_B \cap \mathcal{T}(L)$ that induce (p, K) is an m-dimensional polyhedral cone $\Gamma(p, K)$ in $\mathbb{R}^n$ whose extremal rays can all be taken to be $0/1$ vectors.

Proof by example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

$$p: \quad 5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1 \qquad\qquad K: \quad 1 <_K 4$$

The cyclic Bergman fan

Let K be the restriction of the order J to the image of p. We will say that the pair (p, K) is the **compatible pair** induced by w.

Proposition

Let (p, K) be a compatible pair. The set of vectors $w \in \Sigma_B \cap \mathcal{T}(L)$ that induce (p, K) is an m-dimensional polyhedral cone $\Gamma(p, K)$ in $\mathbb{R}^n$ whose extremal rays can all be taken to be $0/1$ vectors.

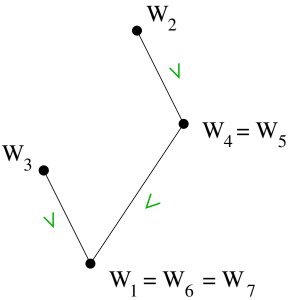
Proof by example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$C(5, B) = \{2, 4, 5\}, \quad C(6, B) = \{1, 2, 4, 6\}, \quad C(7, B) = \{1, 2, 3, 7\}$$

$$p: \quad 5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1 \qquad\qquad K: \quad 1 <_K 4$$

A vector w induces this compatible pair iff it satisfies $\longrightarrow$



The cyclic Bergman fan

Let K be the restriction of the order J to the image of p. We will say that the pair (p, K) is the **compatible pair** induced by w.

Proposition

Let (p, K) be a compatible pair. The set of vectors $w \in \Sigma_B \cap \mathcal{T}(L)$ that induce (p, K) is an m-dimensional polyhedral cone $\Gamma(p, K)$ in $\mathbb{R}^n$ whose extremal rays can all be taken to be $0/1$ vectors.

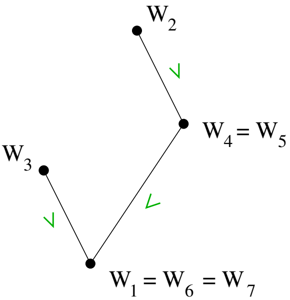
Proof by example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$C(5,B) = \{2, 4, 5\}, \quad C(6,B) = \{1, 2, 4, 6\}, \quad C(7,B) = \{1, 2, 3, 7\}$$

$$p: \quad 5 \mapsto 4, \quad 6 \mapsto 1, \quad 7 \mapsto 1 \qquad\qquad K: \quad 1 <_K 4$$

A vector w induces this compatible pair iff it satisfies $\longrightarrow$



The corresponding cone is generated by the rays

$e_2, \; e_2 + e_4 + e_5, \; e_3, \; \pm(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7)$.

The cyclic Bergman fan

Theorem

The cones $\{\Gamma(p, K) \mid (p, K) \text{ is a compatible pair}\}$ are the maximal cones of a simplicial polyhedral fan whose support is the tropicalization $\mathcal{T}(L)$, called the ***cyclic Bergman fan*** *of L.*

The cyclic Bergman fan

Theorem

*The cones $\{\Gamma(p, K) \mid (p, K)$ is a compatible pair$\}$ are the maximal cones of a simplicial polyhedral fan whose support is the tropicalization $\mathcal{T}(L)$, called the **cyclic Bergman fan** of L.*

Its rays are all vectors of the form $e_F := \sum_{i \in F} e_i$, where F is a flat of A which is either a cyclic flat or a singleton.

The cyclic Bergman fan

Theorem

The cones $\{\Gamma(p, K) \mid (p, K) \text{ is a compatible pair}\}$ are the maximal cones of a simplicial polyhedral fan whose support is the tropicalization $\mathcal{T}(L)$, called the ***cyclic Bergman fan*** *of L.*

Its rays are all vectors of the form $e_F := \sum_{i \in F} e_i$, where F is a flat of A which is either a cyclic flat or a singleton.

We can compute $\mathcal{T}(L)$ by going over all bases B of the matrix A, computing all possible compatible pairs (p, K), and calculating their corresponding cones $\Gamma(p, K)$.

The cyclic Bergman fan

Theorem

*The cones $\{\Gamma(p, K) \mid (p, K)$ is a compatible pair$\}$ are the maximal cones of a simplicial polyhedral fan whose support is the tropicalization $\mathcal{T}(L)$, called the **cyclic Bergman fan** of L.*

Its rays are all vectors of the form $e_F := \sum_{i \in F} e_i$, where F is a flat of A which is either a cyclic flat or a singleton.

We can compute $\mathcal{T}(L)$ by going over all bases B of the matrix A, computing all possible compatible pairs (p, K), and calculating their corresponding cones $\Gamma(p, K)$.

Key Fact

There is an effective algorithm for computing compatible pairs, which builds up both p and K at the same time.

TropLi: A C++ implementation

This algorithm for computing $\mathcal{T}(L)$ has been implemented in C++ as a software tool called TropLi. It can also be used to compute bases, circuits, tutte polynomial...

TropLi: A C++ implementation

This algorithm for computing $\mathcal{T}(L)$ has been implemented in C++ as a software tool called TropLi. It can also be used to compute bases, circuits, tutte polynomial...

Example

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 2 & -2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix A has size 7×14. The software Gfan takes more than a day to to compute $\mathcal{T}(L)$. TropLi takes much less...

TropLi: A C++ implementation

This algorithm for computing $\mathcal{T}(L)$ has been implemented in C++ as a software tool called TropLi. It can also be used to compute bases, circuits, tutte polynomial...

Example

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & -2 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & 0 & 0 & -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 2 & -2 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix A has size 7×14. The software Gfan takes more than a day to to compute $\mathcal{T}(L)$. TropLi takes much less... 1 second!

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -3 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^{\perp} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}$$

The matrix $A^{\perp}$ has size 9×13. A Maple implementation for computing tropical linear spaces locally using their nested fan structure (already much faster than Gfan) takes many hours to compute $\mathcal{T}(L^{\perp})$. TropLi takes...

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -3 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A^\perp = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}$$

The matrix $A^\perp$ has size 9×13. A Maple implementation for computing tropical linear spaces locally using their nested fan structure (already much faster than Gfan) takes many hours to compute $\mathcal{T}(L^\perp)$. TropLi takes... 1 second!

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^{\perp}$ has size 15×20.

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^\perp$ has size 15×20. TropLi computes $\mathcal{T}(L^\perp)$ as fan with 172 rays and 475 722 maximal cones. All the computation takes just a little more than 60 seconds!

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^{\perp}$ has size 15×20. TropLi computes $\mathcal{T}(L^{\perp})$ as fan with 172 rays and 475 722 maximal cones. All the computation takes just a little more than 60 seconds!

A C++ implementation for computing vertices of the Newton polytope of an A-discriminant is also available, at

http://math.berkeley.edu/~felipe/tropli/

TropLi: A C++ implementation

Example

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The matrix $A^\perp$ has size 15×20. TropLi computes $\mathcal{T}(L^\perp)$ as fan with 172 rays and 475 722 maximal cones. All the computation takes just a little more than 60 seconds!

A C++ implementation for computing vertices of the Newton polytope of an A-discriminant is also available, at

http://math.berkeley.edu/~felipe/tropli/

Thank you!

Ooops...