Recent Developments in Weak Convergence Methods for Nonlinear PDE

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Recent Developments in Weak Convergence Methods for Nonlinear PDE

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Key problem:

Approximations – $$\boxed{A_k[u_k] = f_k}$$ $(k=1,2,\dots)$

nonlinear PDE operator

solution of approx. problem

Assume –
- $f_k \to f$
- $A_k[v] \to A[v]$ $\forall$ smooth v
- $u_k \rightharpoonup u$ WEAKLY

Is it true that

$$\boxed{A[u] = f\ ??}$$

[E1]
LCE, *Weak Convergence Methods for Nonlinear PDE*, AMS (1991)

1. Weak convergence
2. Convexity
3. Quasiconvexity
4. Concentrated compactness
5. Compensated compactness
6. Maximum principle methods

Today:

1. Convexity methods
2. Oscillations and cancellation
3. Concentration

Emphasis is not so much on powerful, new estimates, but rather what to do without such estimates.

I. CONVEXITY METHODS

A. Hidden convex structures
B. Quasiconvexity
C. Convex integration

II. OSCILLATIONS & CANCELLATION

A. Homogenization
B. Jacobians
C. Hardy space methods
D. Null forms
E. Commutators

III. CONCENTRATION

A. Defect measures
B. Semiclassical defect measures
C. Microlocal defect measures
D. Other concentration phenomena

Today's topics

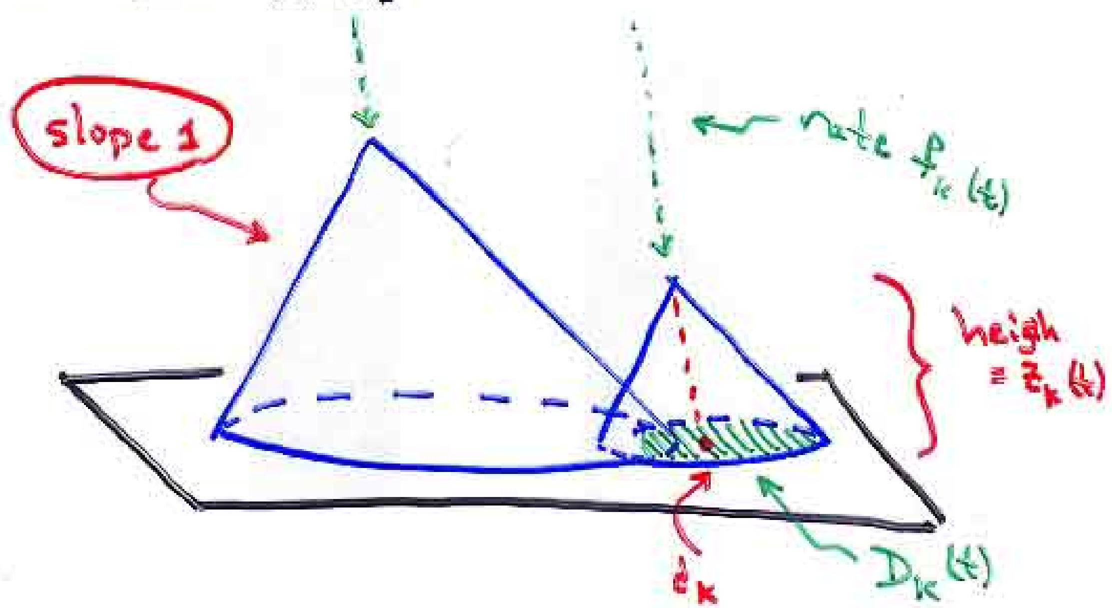
Major omission – Compensated compactness methods for conservation laws –
See survey by G Q Chen [Ch], etc

Topic I: CONVEXITY METHODS

A. Hidden convex structures

- **A first example** – interacting sandcones

Aronsson [Ar]



ODE for heights:

$$\dot{z}_k = \frac{f_k}{|D_k|}$$

area of region when k^{th} cone is tallest

$(k = 1, 2, \cdots)$

What is "hidden" convex structure??

$$I[u] = \begin{cases} 0 & \text{if } |\nabla u| \leq 1 \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases}$$

CONVEX FUNCTION

System of ODE for cone heights

$\Longleftrightarrow$ $\boxed{f - u_t \in \partial I[u]}$

for $f = \sum_{k=1}^{m} f_k(t)\,\delta_{d_k}$

Interacting sandcones = gradient flow in L^2 governed by $\partial I[\cdot]$

[A-E-W], Prigozhin [Pr]

- Monotonicity and Convexity

Notation H = (real) Hilbert space

Def $A : \underbrace{D(A)}_{\subseteq H} \to H$ is monotone

if $\boxed{(x-\hat{x},\, A[x]-A[\hat{x}]) \geq 0}$

$\forall\, x, \hat{x} \in D(A)$

Krylov [Ky]: {See this for details}

$$G(x,y) = \sup_z \{ (x-z, A[z]) + (z,y) \}$$

A max monotone

Then
- G is convex
- $G(x,y) \geq (x,y) \quad \forall x,y$
- $G(x,y) = (x,y) \iff y = A[x]$

Application: $-\mathrm{div}(\underline{B}(\nabla u_k)) = f_k$

$$\left\{ \begin{array}{l} A[x_k] = y_k \\ y_k \to y \\ x_k \rightharpoonup x \end{array} \right. \Longrightarrow A[x] = y$$

(Browder–Minty)

Proof:

$$(x,y) = \lim_{k\to\infty} (x_k, y_k) = \lim_{k\to\infty} G(x_k,y_k)$$

by convexity $\geq G(x,y)$

So $G(x,y) \leq (x,y)$. Thus

$G(x,y) = (x,y) \Rightarrow y = A[x]$

$\square$

Relaxation = Convexification

(best kind of convex problem)

- **Relaxation, linear programming**

Review: $\underbrace{A \in \mathbb{M}^{m\times n}}_{m\times n \text{ matrices}}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$

primal problem

(P) Find $x \in \mathbb{R}^n$ to minimize
$c\cdot x$,
subject to $Ax = b$, $x \geq 0$

dual problem

(D) Find $y \in \mathbb{R}^m$ to maximize
$b\cdot y$,
subject to $A^T y \leq c$.

"Complementary slackness"

If x solves (P) and y solves (D), then

$$c\cdot x = b\cdot y$$

Application 1: Weak KAM theory

(Fathi [F], Mather [Ma])

- **Lagrangian**: $L = L(x,v)$ ← unif. convex in v

  $\mathbb{T}^n$-periodic in x

- **Action of a curve**:

$$A_T[\underline{\gamma}] = \int_0^T L(\underline{\gamma}(t), \dot{\underline{\gamma}}(t))\,dt$$

Original problem: Find $\underline{\gamma} : [0,\infty) \to \mathbb{R}^n$

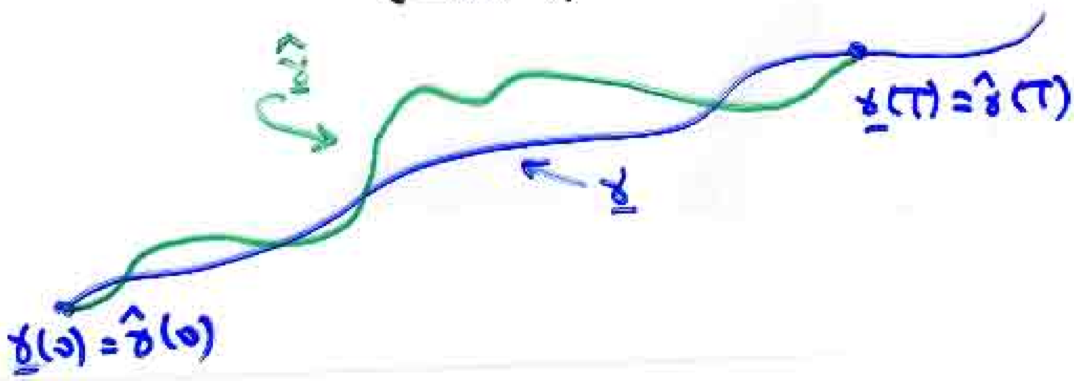
s.t.

$$\boxed{A_T[\underline{\gamma}] \le A_T[\hat{\underline{\gamma}}]}$$

for all $T > 0$ and all $\hat{\underline{\gamma}}(0) = \underline{\gamma}(0)$, $\hat{\underline{\gamma}}(T) = \underline{\gamma}(T)$

and such that

$$\lim_{t\to\infty} \frac{\underline{\gamma}(t)}{t} = V$$ ← given



Relaxed problem: Find probability measure μ to minimize

$$\boxed{A[\mu] = \int_{\mathbb{R}^n}\int_{\mathbb{T}^n} L(x,v)\, d\mu,}$$

subject to $\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} v\cdot\nabla\phi\, d\mu = 0 \quad \forall \phi \in C^1_x$

$$\underbrace{\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} v\, d\mu = V, \quad \mu \geq 0}$$

This is (an infinite dimensional) linear programming problem. Study dual problem, etc.

Forni - Mather [F-M], [E-G1], [E-G2]

Application 2: optimal mass transport

- $f^{\pm} \geq 0, \quad \int f^+ \, dx = \int f^- \, dy$
- $\underline{s} : \mathbb{R}^n \to \mathbb{R}^n, \quad \underbrace{\underline{s}_{\#} f^+ = f^-}$

push forward

$\underline{s}$ rearranges the mass $f^+ dx$ to $f^- dy$

- **Cost** $\quad C[\underline{s}] = \int_{\mathbb{R}^n} \overset{\text{given } c(x,y)}{c(x, \underline{s}(x))} f^+ dx$

Original problem: Find $\underline{s} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\boxed{C[\underline{s}] \le C[\hat{\underline{s}}]}$$

among all maps that rearrange $f^+ dx$ into $f^- dy$

Relaxed problem: Find a nonnegative measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ to minimize

$$\boxed{C[\gamma] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} c(x,y)\, d\gamma}$$

subject to

$$\underbrace{\begin{aligned} \text{proj}_x \gamma &= f^+ dx \\ \text{proj}_y \gamma &= f^- dy \quad \gamma \ge 0 \end{aligned}}$$

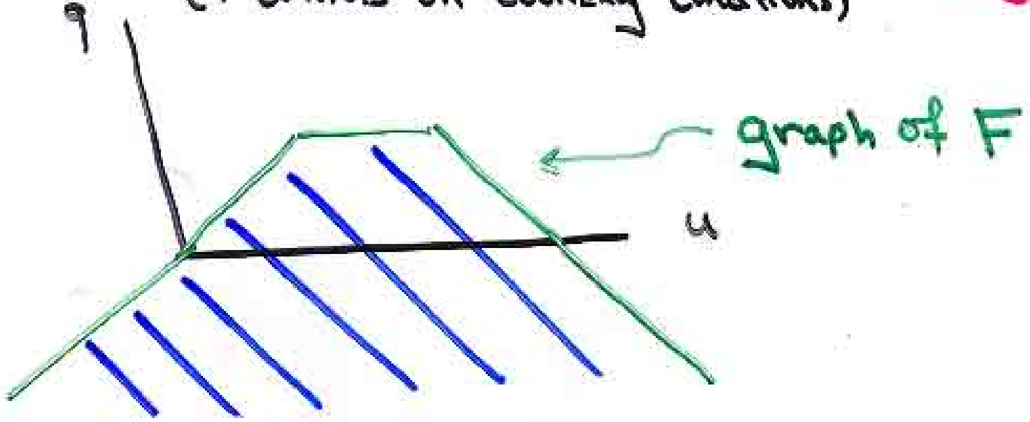
This is (an infinite dimensional) linear programming problem. Study dual problem, etc

See [E 2], Villani [V]

Application 3: Control of traffic flow

- $$u_t + F(u)_x = 0$$ (+ controls on boundary conditions)

  u = traffic density



- Relaxation: $$\begin{cases} u_t + q_x = 0 \\ q \leq F(u) \end{cases}$$

G. Gomes + R. Horowitz, "Optimal freeway ramp metering using asymmetric cell transmission model", to appear.

- Displacement Convexity McCann [Mc]

A nonconvex minimization problem:

$$\mathcal{P} = \{\rho : \mathbb{R}^n \to \mathbb{R} \mid \rho \geq 0, \int_{\mathbb{R}^n} \rho \, dx = 1\}$$

$$E[\rho] = \int_{\mathbb{R}^n} \rho^\gamma \, dx + \underbrace{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho(x)\rho(y) V(x-y) \, dx dy}_{\text{not convex in } \rho}$$

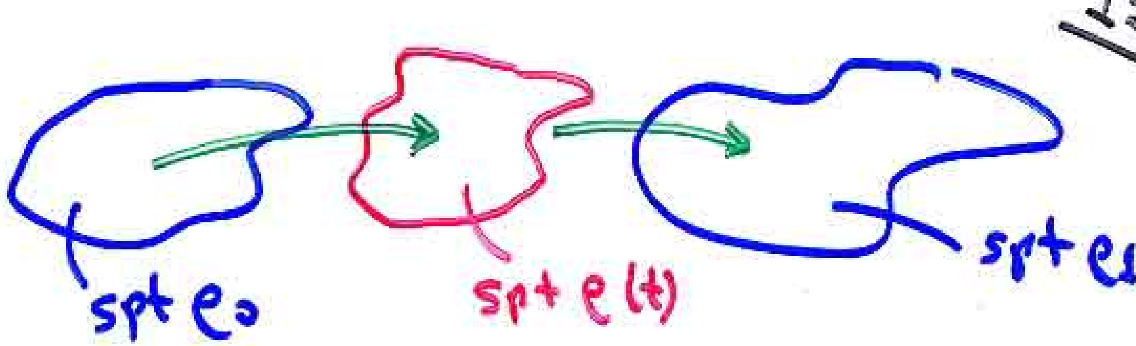
Assume $V \geq 0$, V convex

Mass transfer $c(x,y) = |x-y|^2$

$\underline{s}$ = optimal transfer plan from ρ_0 to ρ_1

Define: $\rho(t) = [(1-t)I + t\underline{s}]_{\#} \rho_0$ $(0 \leq t \leq 1)$

Theorem $\boxed{t \mapsto E[\rho(t)] \text{ is convex for } 0 \leq t \leq 1}$



$E[\cdot]$ is convex along this path

Cf: Villani [V], Jordan–Kinderlehrer–Otto [J-K-O], Otto [Ot], etc

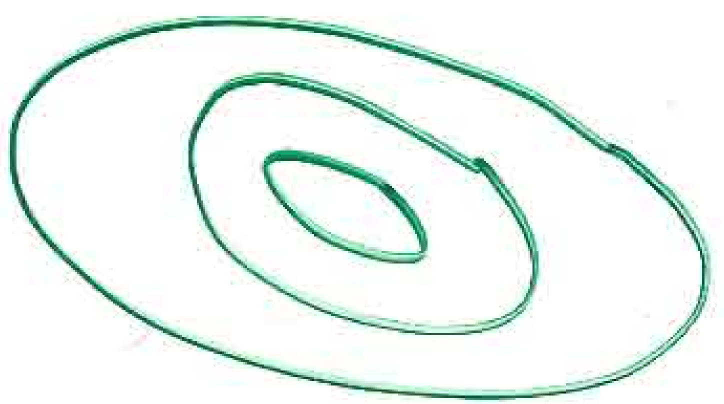
B. Quasiconvexity

← 2 different meanings

- <u>Level set convexity</u>

<u>Def</u> $F : \mathbb{R}^n \to \mathbb{R}$ is level set convex if $\forall c \in \mathbb{R}$,

the set $\{x \mid F(x) \le c\}$ is convex



Level set convexity is called "quasi convexity" in math econ literature

<u>Application</u>

$$A_F[u] = \underbrace{\frac{\partial F}{\partial p_i}(\nabla u)\frac{\partial F}{\partial p_j}(\nabla u)u_{x_i x_j}}_{\text{"Aronsson's equation"}} = 0$$

Barron-Jensen-Wang [B-J-W]:

$\boxed{\text{Boundary value problems for } A_F \text{ are well-posed} \iff F \text{ is level set convex}}$

- <u>Morrey quasi convexity</u>

<u>Def</u> $F: \mathbb{M}^{m \times n} \to \mathbb{R}$ is quasi convex in the sense of Morrey if ...

$$\boxed{F(A) \le \int_{B(0,1)} F(A+\nabla\phi)\,dx}$$

$\forall A \in \mathbb{M}^{m\times n}$, $\forall \phi \in C^1_0(B(0,1))$

New examples, Counterexamples – Sverak

- If $\underline{u}: \mathbb{R}^n \to \mathbb{R}^m$, its <u>energy</u> is

$$I[\underline{u}] = \int_{\mathbb{R}^n} F(\nabla \underline{u})\,dx$$

Dal Maso – Francfort – Toader [DM – F – T]

<u>Theorem</u> (i) If $\underline{u}_k \rightharpoonup \underline{u}$ in H^1, then

$$I[\underline{u}] \le \liminf_{k\to\infty} I[\underline{u}_k]$$

(ii) If $\underline{u}_k \rightharpoonup \underline{u}$ and $I[\underline{u}_k] \to I[\underline{u}]$, then

$$\nabla F(\nabla \underline{u}_k) \longrightarrow \nabla F(\nabla \underline{u})$$

Interesting new observation

<u>Proof</u> $\forall \eta > 0$

$$\underbrace{\int \frac{F(\nabla \underline{u} + \eta \underline{\Psi}) - F(\nabla u)}{\eta}}_{\approx \int \nabla F(\nabla \underline{u}) : \underline{\Psi}} \leq \liminf_{k \to \infty} \underbrace{\int \frac{F(\nabla \underline{u}^k + \eta \underline{\Psi}) - F(\nabla \underline{u}^k)}{\eta}}_{\approx \int \nabla F(\nabla \underline{u}^k) : \underline{\Psi}}$$

So $\displaystyle \int \nabla F(\nabla \underline{u}) : \underline{\Psi} \leq \lim_{k \to \infty} \int \nabla F(\nabla \underline{u}^k) : \underline{\Psi}$

Replace $\underline{\Psi}$ by $-\underline{\Psi}$ $\square$

C. Convex integration

- <u>Differential inclusions</u>

<u>Given</u> $K \subset \mathbb{M}^{m \times n}$, $U \subset \mathbb{R}^n$

$g : \overline{U} \to \mathbb{R}^m$

BASIC PROBLEM:

Find $\underline{u} : \overline{U} \to \mathbb{R}^m$ such that

$$\begin{cases} \nabla \underline{u} \in K & \text{a.e. in } U \\ \underline{u} = \underline{g} & \text{on } \partial U \end{cases}$$

Many PDE can be put into this form

<u>Method of convex integration</u> (Gromov, Müller–Sverak)

(i) Build <u>approximate</u> <u>solutions</u> $\underline{u}^k$ $(k = 1, \ldots)$

(ii) <u>Show</u> $\underline{u}^k \to \underline{u}$ <u>strongly</u> in <u>$W^{1,1}$</u> $(\nabla \underline{u}^k \to \nabla \underline{u}$ a.e.)

<u>Approximation</u>: Given $G \supset K$ and $\underline{v}$ solving

$$\begin{cases} \nabla \underline{v} \in G & \text{a.e. in } U \\ \underline{v} = \underline{g} & \text{on } \partial U, \end{cases}$$

modify $\underline{v}$ to build $\underline{u}^k$ satisfying ...

$$(*)\ \begin{cases} \nabla \underline{u}^k \in \overset{\supset K}{G} \text{ a.e. in } U \\ \| \operatorname{dist}(\nabla u^k, K) \|_{L^1} \to 0 \\ \underline{u}^k = g \text{ on } \partial U, \end{cases}$$

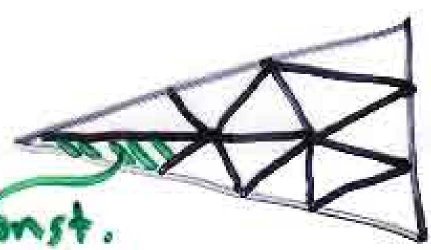
such that $\underbrace{\nabla \underline{u}^k \to \nabla \underline{u} \text{ a.e.}}_{\text{How can we do this??}}$ $\longrightarrow$

See page 17½

- **Building approximations** * Sychev [Sy] *

Approximation hypothesis: $\forall A \in G \ \forall \varepsilon > 0$ $\exists$ a piecewise affine function $\underline{v}$ such that

$$\begin{cases} \nabla v \in G \text{ a.e. in } U \\ \| \operatorname{dist}(\nabla v, K) \|_{L^1} \leq \varepsilon \\ v = Ax \text{ on } \partial U \end{cases}$$



Principle: "Controlled L^∞ convergence of $\underline{u}^k$ implies L^1 convergence of $\nabla \underline{u}^k$"

Kirchheim [Kr] lecture notes,
Zhang [Z2]

But

$$\nabla \underline{u}^k \to \nabla \underline{u} \quad \text{a.e.}$$

if $\|\underline{u}^k - \underline{u}\|_{L^\infty}$ is small compared to oscillations of $\nabla \underline{u}^k$

Theorem Suppose $\underline{g}$ is piecewise affine,

$\nabla \underline{g} \in G$ a.e. in U.

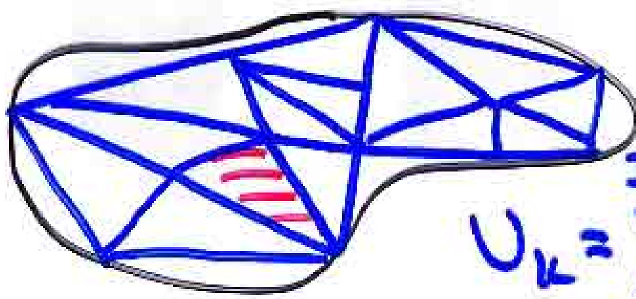
Then $\exists$ $\underline{u}$ such that $G \supsetneq K$

$$\begin{cases} \nabla \underline{u} \in K & \text{a.e. in } U \\ \underline{u} = \underline{g} & \text{on } \partial U \end{cases}$$

Proof We will build $\underline{u}^k$ as above, with

$\underline{u}^k \to \underline{u}$ strongly in $W^{1,1}$

- $\underline{u}^1 := \underline{g}$. Given $\underline{u}^k$, build $\underline{u}^{k+1}$:



- $Du^k \big|_{T_k^i}$ is constant

Apply Approximation hypothesis on T^i_k.

$$\underline{u}^{k+1} = \underline{u}^k + \phi^i_k \quad \text{on } T^i_k)$$

when ϕ^i_k is piecewise affine,

$$\begin{cases} \nabla(\underline{u}^k + \phi^i_k) \in G \\ \| \operatorname{dist}(\nabla \underline{u}^k + \nabla \phi^i_k, K) \|_{L^1(T^i_k)} \leq \frac{1}{2^k} \\ \| \phi^i_k \|_{L^\infty(T^i_k)} \leq \underbrace{s_k}_{\to 0 \text{ fast}} \end{cases}$$

- Check $\underline{u}^k \to \underline{u}$ uniformly

T^i_k

$$\underline{v}^k := \begin{cases} \underline{u} & \text{on vertices of } T^i_k \\ \text{affine} & \text{inside } T^i_k \end{cases}$$

Then $\| \underline{u}^k - \underline{v}^k \|_{L^\infty(T^i_k)}$ is small

$\Longrightarrow$ $\| \nabla \underline{u}^k - \nabla \underline{v}^k \|_{L^\infty(T^i_k)}$ is small,

since $\underline{u}^k, \underline{v}^k$ are affine on T^i_k

Now $\nabla \underline{v}^k \approx \nabla u$ $\square$

Applications (deep) : Müller - Sverak [M-S1]

There exists Lipschitz $\underline{u}$ solving

$$\boxed{\nabla \cdot (\nabla F(\nabla \underline{u})) = 0},$$

F strictly quasiconvex

but $\underline{u} \notin C^1(V)$ for any open set V.

COUNTEREXAMPLE TO PARTIAL REGULARITY for critical points (bad) of $I[\cdot]$

- Baire category methods

Dacorogna - Marcellini [D-M1]

Model problem

$$\begin{cases} F(\nabla u) = 0 & \text{a.e. in } U \\ u = g & \text{on } \partial U \end{cases}$$

Assume: F is convex, $\lim_{|p|\to\infty} F(p) = \infty$,

$\underbrace{g = a\cdot x}_{\text{affine boundary conditions}}$, $F(a) < 0$

$\boxed{\text{Theorem } \exists\ \infty \text{ many Lipschitz solutions}}$

Proof $X = \{u \in C^{0,1} \mid u = g \text{ on } \partial U,\ F(\nabla u) \leq 0 \text{ a.e.}\}$,

with sup-norm topology.

$$V_k := \{u \in X \mid \int_U F(\nabla u)\,dx > -\tfrac{1}{k}\}$$

Claim #1 V_k is open

Proof Let $u_j \in X - V_k$, $u_j \to u$ uniformly

$\nabla u_j \xrightarrow{*} \nabla u$

$$\int F(\nabla u) \leq \liminf \int F(\nabla u_j) \leq -\frac{1}{k}$$

So $u \in \mathbb{X} - V_k$.

Claim #2 V_k is dense (See [D-M I])

Since $\mathbb{X}$ is complete metric space,

$$\bigcap_{k=1}^{\infty} V_k \text{ is dense in } \mathbb{X}.$$

Baire Category Theorem

$$u \in \bigcap_{k=1}^{\infty} V_k \Rightarrow F(\nabla u) = 0 \text{ a.e.}$$

$\square$

The book [D-M I] has many other examples —

- See references for recent work on gradient flows $\dot{\underline{u}} = -\nabla I[\underline{u}]$

Topic II: OSCILLATIONS AND CANCELLATION

A. Homogenization

• Two-scale Convergence

Nguetseng [Ng], Allaire [A], Tartar

Def A sequence $\{u^{\varepsilon_j}\}$ two-scale converges to $u = u(x,y)$ if

$$\boxed{\lim_{\varepsilon_j \to 0} \int_U u^{\varepsilon_j}(x)\, \phi\left(x, \frac{x}{\varepsilon_j}\right) dx = \int_U \int_{\mathbb{T}^n} u(x,y)\, \phi(x,y)\, dy\, dx}$$

$\forall\ \phi \in C^\infty(U \times \mathbb{T}^n)$

We write $u^{\varepsilon_j} \xrightarrow{2} u$

Theory –

- If $\{u^\varepsilon\}$ is bounded in L^2, $\exists$ a subsequence $\{u^{\varepsilon_j}\}$ that 2-scale converges
- Suppose $u^\varepsilon \rightharpoonup u$ in H^1. Then $\exists$ $\tilde{u} \in L^2(U, H^1(\mathbb{T}^n))$ such that

$$\boxed{\nabla u^\varepsilon \xrightarrow{2} \nabla u + \nabla_y \tilde{u}}$$

<u>Application</u>: standard homogenization

$$-\left(a_{ij}\left(\tfrac{x}{\varepsilon}\right) u^\varepsilon_{x_i}\right)_{x_j} = f$$

Multiply by $\phi(x) + \varepsilon \tilde{\phi}\left(x, \tfrac{x}{\varepsilon}\right)$ perturbed test function

$$\int_U a_{ij}\left(\tfrac{x}{\varepsilon}\right) u^\varepsilon_{x_i}\left(\phi_{x_j} + \tilde{\phi}_{y_j} + \varepsilon \tilde{\phi}_{x_j}\right) dx$$

$$= \int_U f\left(\phi + \varepsilon \tilde{\phi}\right) dx$$

Let $\varepsilon = \varepsilon_j \to 0$ . . . .

$$\int_U \int_{\mathbb{T}^n} a_{ij}(y)(u_{x_i} + \tilde{u}_{y_i})(\phi_{x_j} + \tilde{\phi}_{y_j})\,dx\,dy = \int_U f\phi\,dx$$

This holds $\forall\, \phi, \Phi$

- $(a_{ij}(y)(u_{x_i} + \tilde{u}_{y_i}))_{y_j} = 0$
- $\left(\int_{\mathbb{T}^n} a_{ij}(y)(u_{x_i} + \tilde{u}_{y_i})\,dy\right)_{x_j} = f$

Solve these to get usual formulas, etc. □

- See Allaire [A] for homogenization of
$$I_\varepsilon[v] = \int F(Dv, \tfrac{x}{\varepsilon})\,dx$$
- See Cioranescu - Damlamian et al [C-D-G] [C-D-DA] for periodic unfolding method.

- <u>Large oscillations</u>

Capdeboscq [Cp], Allaire - Capdeboscq - Piatnitski - Siess - Vanninathan [A-C-P-S-V]

Model problem:

$$(*) \quad \begin{cases} -\varepsilon^2 \Delta u^\varepsilon + V\left(\frac{x}{\varepsilon}\right) u^\varepsilon = \lambda_0 u^\varepsilon + \underbrace{\varepsilon^2 f}_{\text{perturbation}} \\ \end{cases}$$

periodic

Eigenvalue problem:

$$\begin{cases} -\Delta \phi + V(y)\phi = \lambda_0 \phi & \text{in } \mathbb{T}^n \\ \phi > 0 \end{cases}$$

principal eigenfunction

$\phi^\varepsilon(x) = \phi\left(\frac{x}{\varepsilon}\right)$ solves

$$-\varepsilon^2 \Delta \phi^\varepsilon + V\left(\frac{x}{\varepsilon}\right)\phi^\varepsilon = \lambda_0 \phi^\varepsilon$$

Try $\quad u^\varepsilon(x) = \underbrace{\phi\left(\frac{x}{\varepsilon}\right)}_{\text{large oscillations}} v^\varepsilon(x)$

$$-\cancel{\varepsilon^2}\left((\phi^\varepsilon)^2 v^\varepsilon_{x_i}\right)_{x_i}$$

$$= -\varepsilon^2\left((\phi^\varepsilon)^2 \left(\frac{u^\varepsilon}{\phi^\varepsilon}\right)_{x_i}\right)_{x_i}$$

$$= -\varepsilon^2\left(u^\varepsilon_{x_i}\phi^\varepsilon - u^\varepsilon\phi^\varepsilon_{x_i}\right)_{x_i}$$

$$= -\varepsilon^2\left(\Delta u^\varepsilon\,\phi^\varepsilon - u^\varepsilon\,\Delta\phi^\varepsilon\right)$$

$$= \cancel{\varepsilon^2} f\,\phi^\varepsilon$$

So

$$\underbrace{-\left(\phi\left(\tfrac{x}{\varepsilon}\right)^2 v^\varepsilon_{x_i}\right)_{x_i} = f\,\phi\left(\tfrac{x}{\varepsilon}\right)}_{\text{Routine to find PDE solved by } v = \lim_{\varepsilon\to 0} v^\varepsilon}$$

<u>Open question</u>: fully nonlinear versions?

<u>Stochastic</u> <u>homogenization</u> – many interesting questions

What do the effective equations "mean"?

B. Jacobians

Key identities:

- $A^T \operatorname{cof} A = (\det A) I$
- $\operatorname{div}(\operatorname{cof} \nabla \underline{u}) \equiv 0$

Use these to understand effects of oscillation / cancellation

Reference: book by Iwaniec & Martin [I-M 2]

- Mappings with gradients in $SO(n)$

$AA^T = I$, $\det A = 1$

Reshetnyak's Thm Suppose $\underline{u}: \mathbb{R}^n \to \mathbb{R}^n$, $\nabla \underline{u} \in SO(n)$. Then

$$\underline{u}(x) = Ax + a$$

Proof $(\nabla \underline{u})^T \operatorname{cof} \nabla \underline{u} = (\det \nabla \underline{u}) I = I$

$$(\nabla \underline{u})^T \nabla \underline{u} = I$$

$$\Longrightarrow \quad \nabla \underline{u} = \operatorname{cof} \nabla \underline{u}$$

$$\therefore \quad 0 = \operatorname{div}(\operatorname{cof} \nabla \underline{u}) = \operatorname{div}(\nabla \underline{u}) = \Delta \underline{u}$$

Also,

$$\Delta\left(|\nabla \underline{u}|^2 - n\right) = 2\nabla \underline{u} \cdot \nabla \underbrace{\Delta \underline{u}}_{=0} + 2|D^2 \underline{u}|^2$$

and

$$|\nabla \underline{u}|^2 = \operatorname{tr}(\nabla \underline{u}^T \nabla \underline{u}) = n$$

$$\therefore \quad |D^2 \underline{u}|^2 \equiv 0.$$

$\square$

Plate theories (= Γ-limit of 3-D elasticity models)

Friesecke - James - Müller [F-J-M]

Key estimate:

$$\boxed{\|\nabla \underline{u} - R\|_{L^2} \leq C \|\operatorname{dist}(\nabla \underline{u}, SO(n))\|_{L^2} \quad \text{for some } R \in SO(n)}$$

C. Hardy space methods

Reference: Stein's book [St], Chapters 3 & 4

- **Definitions, theory**

Let $\phi \in C_c^\infty(B(0,1))$, $\int \phi \, dx = 1$

- $g^*(x) = \sup_{r>0} \left| \fint_{B(x,r)} g(y) \, \phi\left(\frac{y-x}{r}\right) dy \right|$

- $\|g\|_{\underbrace{\mathcal{H}^1}_{\text{Hardy space}}} = \|g^*\|_{L^1}$

$(\mathcal{H}^1)^* = BMO$ (C. Fefferman)

$\mathcal{H}^1$ is a "replacement" for L^1
BMO ——— " ——— L^∞

- **Div - Curl methods**

Coifman - Lions - Meyer - Semmes [C-L-M-S]

$$\boxed{\begin{array}{l} \underline{\text{Theorem}} \text{ Suppose } \underline{a} \in L^2,\ w \in H^1 \\ \qquad \nabla \cdot \underline{a} = 0. \\ \text{Then} \qquad \underbrace{\underline{a} \cdot \nabla w}_{\text{obviously } \in L^1} \in \mathcal{H}^1 \end{array}}$$

Proof: $\phi_r = \phi\left(\frac{y-x}{r}\right)$

average over $B(x,r)$

$$\int_{B(x,r)} \underline{a} \cdot \nabla w \, \phi_r \, dy = -\int_{B(x,r)} (w - (w)_{x,r})\, \underline{a} \cdot \nabla \phi_r \, dy$$

So

$$\left| \fint_{B(x,r)} \underline{a}\cdot\nabla w \, \phi_r \, dy \right| \le \frac{C}{r^{n+1}} \int_{B(x,r)} |w-(w)_{x,r}| \, |\underline{a}$$

$$\le C \left(\fint_{B(x,r)} |\nabla w|^s dx \right)^{1/s} \left(\fint_{B(x,r)} |a|^t dy \right)^{1/t}$$

$$\le \underbrace{C \left[M(|\nabla w|^s)^{2/s} + M(|a|^t)^{2/t} \right]}_{\in L^1}$$

maximal functions

$\square$

Application #1 : Wente's Lemma

$$\begin{cases} -\Delta u = f_x g_y - f_y g_x \\ f, g \in H^1(\mathbb{R}^2). \end{cases} \qquad (n=2)$$

Thm

$$\boxed{\|u\|_{L^\infty} \le C \left(\|f\|_{H^1}^2 + \|g\|_{H^1}^2 \right)}$$

Proof $f_x g_y - f_y g_x = \underline{a}\cdot\nabla w$

for $w = f$, $\underline{a} = (g_y, -g_x)$ $\quad \nabla\cdot\underline{a} = 0$

So $f_x g_y - f_y g_x \in \mathcal{H}^1$.

$\Phi = \log r \in BMO$

$\Rightarrow \quad u = \Phi * (f_x g_y - f_y g_x) \in L^\infty$

Application #2 : Harmonic maps into spheres

$$\begin{cases} -\Delta \underline{u} = |\nabla \underline{u}|^2 \underline{u} \\ |\underline{u}| = 1 \end{cases} \qquad (n \geq 2)$$

$\underline{u} \in H^1$.

Theorem $\boxed{|\nabla \underline{u}|^2 \underline{u} \in \mathcal{H}^1}$

Proof $|\underline{u}|^2 \equiv 1 \longrightarrow u^j u^j_{x_k} \equiv 0$

Helein

$$\begin{aligned} |\nabla \underline{u}|^2 u^i &= u^j_{x_k} u^j_{x_k} u^i \\ &= u^j_{x_k} (u^j_{x_k} u^i - u^j u^i_{x_k}) \\ &= u^j_{x_k} a^k_{ij} \end{aligned}$$

$$(a^k_{ij})_{x_k} = (\Delta u^j u^i - \Delta u^i u^j)$$
$$= -|\nabla \underline{u}|^2 (u^i u^j - u^i u^j) = 0$$

$\square$

Consequences:

(n=2) $\underline{u}$ is continuous, ∴ smooth

(n≥3) partial regularity for stationary harmonic maps

[E S], Bethuel [Be], Helein's book [H]

(n≥3) biharmonic maps

Chang-Wang-Yang [C-W-Y]

* Rivière* [Rv], Rivière-Struwe [Rv-S]:

$$\Delta \underline{u} = \underbrace{\Omega}_{\text{anti-symmetric}} : \nabla \underline{u}$$

+ gauge transformations

Application #3 : Navier-Stokes equations

$$\begin{cases} \underline{u}_t + (\underline{u}\cdot\nabla)\underline{u} - \nu\Delta\underline{u} = -\nabla p \\ \nabla\cdot\underline{u} = 0 \end{cases}$$

See PL Lions' books on fluid mechanics, Vol I [L5]:

$$\boxed{D^2 p \in L^1(0,T,\mathcal{H}^1)}$$

Proof $\quad -\Delta p = u^i_{x_j} u^j_{x_i} \in \mathcal{H}^1$

Application #4 : Geometric problems $(n=2)$

Sverak-Müller [M-S2]

Strzelecki [S7]

$$\Delta\underline{u} = 2H(\underline{u})(\underline{u}_x \times \underline{u}_y)$$

More "div-curl" methods:

Bourgain-Brezis [B-B1] [B-B2]
Van Schaftingen [VS]

$\nabla \cdot \underline{f} = 0$

$$\begin{cases} \nabla \times \underline{z} = \underline{f} \\ \nabla \cdot \underline{z} = 0 \end{cases}$$

$\underline{z} = \Delta^{-1}(\nabla \times \underline{f})$

$(n=3)$

$$\|\underline{z}\|_{L^{3/2}} \leq C \|\underline{f}\|_{L^1}$$

- <u>Nonnegative vorticity</u> Delort [De]

E-Müller [E-M], Semmes [S]

<u>Theorem</u> $u \in H^1,$ $\boxed{\omega \geq 0}$ $(n=2)$

$-\Delta u = \omega$

then $\boxed{u_x u_y,\ u_x^2 - u_y^2 \in \mathcal{H}^1}$

Interpretation

ω = vorticity

$\underline{v} = \nabla u^{\perp}$ = velocity

$$v^1 v^2,\ (v^1)^2 - (v^2)^2 \in \mathcal{H}^1$$

DiPerna-Majda rewrote 2-D Euler equations in terms of $v^1 v^2$, $(v^1)^2 - (v^2)^2$

New paper by Tom Hou - Delort

D. Null forms

Klainerman, Klainerman-Machedon [K-M]
Shatah-Struwe [S-S], Sogge [Sg]

$$\Box \underline{u} = \underline{u}_{tt} - \Delta \underline{u} = \underbrace{Q(\nabla \underline{u}, \nabla \underline{u})}_{\substack{\text{quadratic in } \nabla \underline{u}, \\ \text{"Jacobian" structure}}}$$

Tools: Klainerman-Sobolev $\leq$, Fourier analysis

• Passing to weak limits

Wave maps into spheres:

$$\begin{cases} \Box \underline{u} = (|\nabla_x \underline{u}|^2 - |\underline{u}_t|^2)\,\underline{u} \\ |\underline{u}| = 1 \end{cases}$$

Approximation $\quad \Box \underline{u}^\varepsilon + \frac{1}{\varepsilon}(|\underline{u}^\varepsilon|^2 - 1)\,\underline{u}^\varepsilon = 0$

Weak Convergence : $\nabla \underline{u}^\varepsilon \rightharpoonup \nabla \underline{u}$ in L^2

$$(\Box u^{\varepsilon,i})\,u^{\varepsilon,j} - (\Box u^{\varepsilon,j})\,u^{\varepsilon,i} = 0$$

$$\parallel$$

$$(u^{\varepsilon,i}_t\, u^{\varepsilon,j})_t - (u^{\varepsilon,i}_{x_k}\, u^{\varepsilon,j})_{x_k}$$
$$- (u^{\varepsilon,j}_t\, u^{\varepsilon,i})_t + (u^{\varepsilon,j}_{x_k}\, u^{\varepsilon,i})_{x_k}$$

Pass to weak limits:

$$(\Box u^i)\,u^j - (\Box u^j)\,u^i = 0$$

$\Box \underline{u}$ is $\parallel$ to $\underline{u}$ $\qquad \Box \underline{u} = \lambda \underline{u}$ $\qquad \lambda = |\nabla_x \underline{u}|^2 - |\underline{u}_t|^2$

- Estimates for null forms

Vectorfields:

rotations
$$\begin{cases} \Omega_{jk} = x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \\ \Omega_j = x_j \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_j} \end{cases}$$

dilation $S = t \frac{\partial}{\partial t} + x_k \frac{\partial}{\partial x_k}$

translations
$$\begin{cases} T_0 = \frac{\partial}{\partial t} \\ T_k = \frac{\partial}{\partial x_k} \end{cases}$$

conformal
$$\begin{cases} K_0 = (|x|^2 - t^2) \frac{\partial}{\partial t} - 2tS \\ K_i = (|x|^2 - t^2) \frac{\partial}{\partial x_i} - 2x_i S \end{cases}$$

Recall $[X, Y] = XY - YX$

$$\left.\begin{aligned} Q(u,v) &= u_t v_t - \nabla_x u \cdot \nabla_x v \\ Q_{k\ell}(u,v) &= u_{x_k} v_{x_\ell} - u_{x_\ell} v_{x_k} \end{aligned}\right\} \text{null forms}$$

$$[\Gamma, Q](u,v) = \Gamma\, Q(u,v) - Q(\Gamma u, v) - Q(u, \Gamma v)$$

vector field

Study:

$$[\Omega_{k\ell}, Q] = 0$$

$$[S, Q_{k\ell}] = 0$$

$$[S, Q] = -2Q$$

$$[\Omega_{k\ell}, Q_{ij}] = \ldots.$$

See references for more details....

<u>Applications</u>: Local well posedness of

$$\Box \underline{u} = Q$$

for less regular initial data, ...

E. Commutators

- Basic idea

- Let $Tu(x) = \int \underbrace{K(x-y)}_{\text{singular C-Z kernel}} u(y)\,dy$

Eg: $T = \frac{\partial^2}{\partial x_i \partial x_j}(-\Delta)^{-1}$

- Given a function ϕ

- Look at

$$\boxed{\underbrace{[\phi, T]}_{\text{Commutator}} u(x)} = \phi(Tu) - T(\phi u)$$

$$= \int K(x-y)\underbrace{(\phi(x)-\phi(y))}_{\text{Use this term to "cancel" part of singularity of } K} u(y)\,dy$$

Coifman, Coifman-Meyer [C-M]

(See also Feireisl [Fe])

Application: PL Lions [L2], Vol 2.

Compressible Navier-Stokes equation

$$\begin{cases} \rho_t + \nabla\cdot(\rho \underline{u}) = 0 \\ (\rho\underline{u})_t + \nabla\cdot(\rho\,\underline{u}\otimes\underline{u}) - \mu\,\Delta\underline{u} \\ \qquad - \eta\nabla(\nabla\cdot\underline{u}) + \nabla\rho^\gamma = 0 \end{cases}$$

p = pressure $= \rho^\gamma$

ρ = density

$\underline{u}$ = velocity

$\mu > 0$

$\mu + \eta > 0$

Convergence of sol'ns Let $(\rho^k, \underline{u}^k)$ be a sequence of solutions.

- $$\sup_{0\le t\le T} \int \rho^k |\underline{u}^k|^2 + (\rho^k)^\gamma \, dx + \int_0^T\!\!\int |\nabla \underline{u}^k|^2 \, dx\, dt \le C$$

- $$\|\rho^k\|_{L^q(\mathbb{T}^n\times(0,T))} \le C \qquad (q > \gamma)$$

Weak limits:

$$\left\{\begin{array}{lll} \rho^k \rightharpoonup \rho & \text{in } L^\infty(0,T,L^\gamma) \\ \underline{u}^k \rightharpoonup \underline{u} & \text{in } L^2(0,T,H^1) \\ \sqrt{\rho^k} \rightharpoonup r & \cdots \\ \sqrt{\rho^k}\,\underline{u}^k \rightharpoonup \underline{v} & \cdots \\ \rho^k \underline{u}^k \rightharpoonup \underline{m} & \cdots \\ \rho^k \underline{u}^{k,i} \underline{u}^{k,j} \rightharpoonup e_{ij} & \cdots \end{array}\right.$$

and

$$(\rho^k)^\gamma \longrightarrow p$$

Theorem (i) $\underline{v} = r\underline{u}$, $\underline{m} = \rho\underline{u}$,
$e_{ij} = \rho u^i u^j$

(ii) If $\rho_0^k \to \rho_0$ in L^1, then
$\rho^k \to \rho$ in L^r $(1 \leq r < q)$

and $(\rho, \underline{u})$ solves Compressible N–S

$p = \rho^\gamma$

Main ideas of proof:

- **Lemma** Let $\boxed{\begin{matrix} g^k \rightharpoonup g \\ h^k \rightharpoonup h \end{matrix}}$ $\begin{matrix} \text{in } L^q_t L^p_x \\ \text{in } L^{q'}_t L^{p'}_x \end{matrix}$

Control in t (i) $\{g^k_t\}$ is bdd in $L^1(0,T, W^{-m,1})$

Control in x (ii) $\|h^k(\cdot,t) - h^k(\cdot + \xi, t)\|_{L^{q'}_t L^{p'}_x} = o(1)$ as $\xi \to 0$

Then $g^k h^k \rightharpoonup gh$ in $\mathcal{D}'$

- Use Lemma to prove (i):

$$\begin{cases} \rho_t + \nabla \cdot (\rho \underline{u}) = 0 \\ (\rho \underline{u})_t + \nabla \cdot (\rho\, \underline{u} \otimes \underline{u}) - \mu \Delta \underline{u} \\ \qquad - \eta \nabla(\nabla \cdot \underline{u}) + \nabla p = 0 \end{cases}$$

Key question – Is $p = \rho^\gamma$?

Recall p = weak limit of $(\rho^k)^\gamma$

- Let $\begin{cases} s = e \log e \\ s^k = e^k \log e^k \rightharpoonup \bar{s} \\ (e^k)^{\gamma+1} \rightharpoonup \pi \end{cases}$ $\quad e^k \rightharpoonup e$

- Use Commutator smoothing effects to show

$$\boxed{(\bar{s}-s)_t + \nabla\cdot(\underline{u}(\bar{s}-s)) + \frac{1}{\mu+\eta}(\pi - pe) = 0}$$

- Observe that $\pi \geq pe$, $\bar{s} \geq s$

So $$\frac{d}{dt}\int \underbrace{\bar{s}-s}_{\geq 0}\,dx \leq 0$$

Hence $\bar{s} \equiv s$ if $\bar{s}^0 = s^0$.

Thus $e^k \to e$ strongly

$\square$

Topic III: CONCENTRATION

A. Defect measures

• **Definitions**

Suppose $u^k \rightharpoonup u$ weakly in L^2

Definition (i) $\nu_k(E) := \int_E |u^k - u|^2 \, dx$

(ii) If $\nu_k \rightharpoonup \nu$ as measures, we call ν a defect measure for the weak convergence $u^k \rightharpoonup u$

Facts • $\nu \equiv 0 \iff u^k \to u$ strongly in L^2

• $|u^k|^2 dx \rightharpoonup |u|^2 dx + \nu$

Proof $\int_E |u^k - u|^2 dx = \int_E |u^k|^2 - 2u^k u + |u|^2 dx$

So $\lim_{k\to\infty} \int_E |u^k|^2 dx = \int_E |u|^2 dx + \underbrace{\lim_{k\to\infty} \int_E |u^k - u|^2 dx}_{\nu(E)}$

$\square$

Applications to PDE : Study properties of ν when u^k solve certain PDE

$\boxed{\text{Time-dependent problems}}$: Derive an evolution equation for ν. Or differential inequality.

(Example: $\frac{d}{dt}\left(\int \bar{s} - s \, dx\right) \leq 0$)

$\boxed{\text{Static problems}}$: Derive an equation (or variational principle) for ν

• • • •

EXAMPLE Stationary harmonic maps

$$\underline{u} : U \to \underbrace{N}_{\text{target}} \subset \mathbb{R}^m$$

- $\underline{u}$ is weakly harmonic if $\Delta \underline{u} \perp \underbrace{T_{\underline{u}} N}_{\text{tangent space of } N \text{ at } \underline{u}}$

  $\underline{u} \in H^1$

- $\underline{u}$ is stationary if

$$\int_U \nabla\cdot\underline{\phi}\,|\nabla \underline{u}|^2 - 2\phi^\alpha_{x_\beta}\,\underline{u}_{x_\alpha}\cdot\underline{u}_{x_\beta}\,dx = 0$$

$\forall\, \underline{\phi} \in C^1_c$

F-H Lin, Li-Tian, Shond:

Thm Let $\underline{u}^k$ be stationary, $\underline{u}^k \rightharpoonup \underline{u}$ in H^1

(i) $|\nabla \underline{u}^k|^2 dx \rightharpoonup |\nabla \underline{u}|^2 dx + \underbrace{\nu}_{\text{defect measure}}$

(ii) $$\boxed{\int_U \operatorname{div}_\Sigma \underline{\phi}\; d\nu} + \int_U \nabla\cdot\underline{\phi}\,|\nabla u|^2 - 2\phi^\alpha_{x_\beta}\,\underline{u}_{x_\alpha}\cdot\underline{u}_{x_\beta}\,dx = 0$$

When Σ is countably $(n-2)$ rectifiable

$$\nu << \underbrace{\mathcal{H}^{n-2}}_{\text{Hausdorff measure}} \llcorner \Sigma$$

If $\underline{u}$ is stationary, then

$$\int_{U} \operatorname{div}_{\Sigma} \phi \, d\nu = 0$$

See also Lin-Wang [L-W], Li-Tian [L-T], etc.

** Wonderful paper by R. Moser [M] on stationary measures. **

B. Semiclassical defect measures

to study concentrations / oscillations with a known small length scale $\varepsilon > 0$

Notation $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$

$$S^0 = \{ \underbrace{a(x,\xi)}_{\text{symbol}} \mid |D_x^\alpha D_\xi^\beta a| \le C_{\alpha\beta} \langle \xi \rangle^{-|\beta|} \}$$

Def The Weyl quantization of a is the operator

$$\boxed{a^W(x,\varepsilon D)f = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \varepsilon\xi\right) e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi}$$

$a^W : L^2 \to L^2$ is bounded

Def The Wigner transform of u is the function

$$\boxed{W^\varepsilon(u)(x,\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} u\left(x - \frac{\varepsilon v}{2}\right) \bar{u}\left(x + \frac{\varepsilon v}{2}\right) e^{i v\cdot\xi}\, dv}$$

Identity:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W^\varepsilon(u)\, a \, dx\, d\xi = \int_{\mathbb{R}^n} \bar{u} \; a^w(x, \varepsilon D)\, u \, dx$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\ \text{If } \{u^\varepsilon\} \text{ is bounded in } L^2,\ \exists\ \varepsilon_j \to 0 \text{ and a measure } \mu \geq 0 \text{ such that} \\[1ex] \displaystyle\int_{\mathbb{R}^n} \bar{u}^{\varepsilon_j}\, a^w(x, \varepsilon_j D)\, u^{\varepsilon_j}\, dx \longrightarrow \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} a \, d\mu \\[1ex] \text{for all symbols } a \in S^0\end{array}}$$

We call μ a **semiclassical defect measure** (or **Wigner measure**)

Lions-Paul [L-P], Gerard-Markowich-Mauser-Poupaud [G-M-M-P], E-Zworski [E-Z], etc

<u>Goals</u> Suppose u^ε solves a linear PDE involving a known small scale $\varepsilon > 0$. Show <u>localization</u> and <u>transport</u> properties of μ:

- If $\|a^w(x,\varepsilon D)u^\varepsilon\|_{L^2} = o(1)$, $\|u^\varepsilon\|_{L^2} = 1$,

then $\boxed{\operatorname{spt}\mu \subseteq \{a = 0\}}$ localization

- If $\|a^w(x,\varepsilon D)u^\varepsilon\|_{L^2} = o(\varepsilon)$,

then $\boxed{\int_{\mathbb{R}^{2n}} \{a, b\}\, d\mu = 0}$ invariance

for all $b \in C_c^\infty$

$$\{a,b\} = \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b$$

* See especially papers by P. Gérard *

C. Microlocal defect measures

(aka "H-measures")

L. Tartar, P. Gerard (← see references)

to study concentration / oscillation when we do not have a known small length scale

Good review paper by G. Francfort [Fr]

<u>Notation</u> $\underbrace{S_h}_{\text{symbols homogeneous of degree 0.}} = \{ a \in S \mid a(x, \lambda\xi) = a(x,\xi) \}$

<u>Theorem</u> Suppose $u^\varepsilon \rightharpoonup 0$ in L^2. Then $\exists\ \varepsilon_j \to 0$ and a measure $\mu \geq 0$ on $\mathbb{R}^n_x \times S^{n-1}_\xi$ such that

$$\int_{\mathbb{R}^n} \bar{u}^{\varepsilon_j}\, a(x,D)\, u^{\varepsilon_j}\, dx \to \int_{\mathbb{R}^n} \int_{S^{n-1}} a\, d\mu$$

for all $a \in S_h$

Goals: Show localization and transport properties of μ — See references ...

D. Other Concentration phenomena

See references ...

A. Soliton dynamics

Bronski – Jerrard

(Nonlinear Schrödinger equation $\Longrightarrow$ classical dynamics)

B. Renormalized solutions

Lions – DiPerna

C. Critical nonlinear wave equation

Grillakis,
Shatah – Struwe

D. Yamabe-type problems

$\overbrace{\text{E. Landau-Ginzburg problems}}^{\text{huge literature}}$

$$I_\varepsilon[u] = \int \varepsilon \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon}(1-|u|^2)^2\,dx$$

- Static problems

Bethuel-Brezis-Helein book

- Dynamic problems

E-Soner-Souganidis
Bethuel-Orlandi-Smets
Jerrard-Soner ....

$\boxed{\text{PLEASE TELL ME ABOUT OTHER INTERESTING WORK}}$