On the Diagonalization of Quadratic Forms

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Introduction

An undergraduate course in linear algebra sometimes includes a treatment of the elementary theory of quadratic forms. This is not surprising since a quadratic form (over a field F of characteristic not 2) is essentially the same as a symmetric bilinear form, which is in turn the same as a symmetric matrix. The main theorem for quadratic forms proved in a typical linear algebra course is that any quadratic form $q(x_1, \ldots, x_n)$ can be "diagonalized", i.e., after a linear change of variables $\{x_1, \ldots, x_n\} \to \{y_1, \ldots, y_n\}$, q can be written as $\sum_{i=1}^n a_i y_i^2$. Here, the a_i 's are elements of F, some of which may be equal to zero. The *rank* of q is defined to be the number of nonzero a_i 's, and (in case $F = \mathbb{R}$) the *signature* of q is defined to be $r - s$, where r and s are respectively the number of positive and negative a_i 's. Both the rank and the signature depend only on the isometry class of the quadratic form q (and does not depend on the particular diagonalization taken); see, e.g. [1: §5.3], [3: Ch. 9].

While most textbooks offer exercises for the diagonalization of quadratic forms in a small number of variables (say $n \le 5$), there seem to be few good examples for the diagonalization of quadratic forms in n variables. In teaching a course in quadratic form theory, I recently came across the following four explicit n-ary forms:

$$q_1(\mathbf{x}) = \sum_{i,j=1}^n \min\{i,\, j\}\, x_i x_j\,, \qquad q_2(\mathbf{x}) = \sum_{i,j=1}^n \max\{i,\, j\}\, x_i x_j\,,$$

$$q_3(\mathbf{x}) = \sum_{i,j=1}^n (i+j) x_i x_j\,, \qquad q_4(\mathbf{x}) = \sum_{i,j=1}^n |i-j|\, x_i x_j\,.$$

For $n = 4$ (for example), these quadratic forms have the following symmetric matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

The diagonalization for the quadratic forms q_i above turns out to be an interesting exercise, the solution of which also leads to some nontrivial conclusions about the forms themselves. I shall offer my solution(s) below as an illustration for the general process of diagonalizing quadratic forms, for those colleagues who may want to present some challenging examples in a linear algebra class.

Throughout this note, we work over the field of rational numbers, although our main computations are valid over any field of characteristic not 2.

The Forms $q_1,\ q_2$

We start with the form q_1. To diagonalize it, note that

$$\begin{aligned} q_1(\mathbf{x}) &= x_1^2 + 2x_1(x_2+\cdots+x_n) + \textstyle\sum_{i,j=2}^n \min\{i,\ j\}\, x_i x_j \\ &= (x_1+x_2+\cdots+x_n)^2 - (x_2+\cdots+x_n)^2 + 2x_2^2 + 4x_2(x_3+\cdots+x_n) \\ &\qquad + \textstyle\sum_{i,j=3}^n \min\{i,\ j\}\, x_i x_j \\ &= (x_1+\cdots+x_n)^2 + (x_2+\cdots+x_n)^2 - 2(x_3+\cdots+x_n)^2 \\ &\qquad + \textstyle\sum_{i,j=3}^n \min\{i,\ j\}\, x_i x_j. \end{aligned}$$

Repeating this process to the end, we get

$$\begin{aligned} q_1(\mathbf{x}) &= (x_1+x_2+\cdots+x_n)^2 + (x_2+\cdots+x_n)^2 + \cdots + (x_{n-1}+x_n)^2 \\ &\qquad -(n-1)x_n^2 + nx_n^2 \\ &= (x_1+x_2+\cdots+x_n)^2 + (x_2+\cdots+x_n)^2 + \cdots + (x_{n-1}+x_n)^2 + x_n^2. \end{aligned}$$

Of course, once we derived this equation, it can also be checked directly by comparing the coefficients of $x_i x_j$ on the two sides. It follows that q_1 has rank n and signature n. The above sum-of-squares expression for q_1 has appeared in [2]. The fact that q_1 is positive definite was used in [2] in an ingenious way to show that a certain representation ring of a cyclic p-group has no nonzero nilpotent elements.

For the form q_2, we start the diagonalization process from the last variable:

$$\begin{aligned} q_2(\mathbf{x}) &= nx_n^2 + 2nx_n(x_1+\cdots+x_{n-1}) + \textstyle\sum_{i,j=1}^{n-1} \max\{i,\ j\}\, x_i x_j \\ &= n(x_1+\cdots+x_n)^2 - n(x_1+\cdots+x_{n-1})^2 \\ &\qquad +(n-1)(x_1+\cdots+x_{n-1})^2 - (n-1)(x_1+\cdots+x_{n-2})^2 \\ &\qquad + \textstyle\sum_{i,j=1}^{n-2} \max\{i,\ j\}\, x_i x_j \\ &= n(x_1+\cdots+x_n)^2 - (x_1+\cdots+x_{n-1})^2 - (n-1)(x_1+\cdots+x_{n-2})^2 \\ &\qquad + \textstyle\sum_{i,j=1}^{n-2} \max\{i,\ j\}\, x_i x_j\,. \end{aligned}$$

Repeating this process to the end, we arrive at the diagonalization

$$\begin{aligned} q_2(\mathbf{x}) &= n(x_1+\cdots+x_n)^2 - (x_1+\cdots+x_{n-1})^2 - \cdots - (x_1+x_2)^2 - 2x_1^2 + x_1^2 \\ &= n(x_1+\cdots+x_n)^2 - (x_1+\cdots+x_{n-1})^2 - \cdots - (x_1+x_2)^2 - x_1^2\,, \end{aligned}$$

which, again, can be checked directly by comparing coefficients. In particular, q_2 has rank n and signature $1-(n-1) = 2-n$. Here, we get as a bonus a criterion for $q_2(x_1,\ldots,x_n)$ to be nonnegative for real numbers $x_1,\ldots,x_n$, namely:

$$n(x_1+\cdots+x_n)^2 \geq (x_1+\cdots+x_{n-1})^2 + (x_1+\cdots+x_{n-2})^2 + \cdots + x_1^2.$$

An alternative way to obtain a diagonalization for q_2 is to use the following interesting "reciprocal" relation between q_1 and q_2:

$$(*) \qquad q_2(x_1,\ldots,x_n)+q_1(x_n,\ldots,x_1)=(n+1)(x_1+\cdots+x_n)^2.$$

This can be proved either by using the explicit forms of the symmetric matrices for the two forms on the left-hand-side, or by relabelling the variables backwards in $q_2(\mathbf{x})$ and using the max-min relation:

$$\max\{n-i+1,\, n-j+1\}=n+1-\min\{i,\, j\}.$$

Given the equation $(*)$, we get

$$\begin{aligned} q_2(x_1,\ldots,x_n) &= (n+1)(x_1+\cdots+x_n)^2-q_1(x_n,\ldots,x_1)\\ &= (n+1)(x_1+\cdots+x_n)^2-\big((x_n+\cdots+x_1)^2+(x_{n-1}+\cdots+x_1)^2\\ &\qquad +\cdots+(x_2+x_1)^2+x_1^2\big)\\ &= n(x_1+\cdots+x_n)^2-(x_1+\cdots+x_{n-1})^2-\cdots-(x_1+x_2)^2-x_1^2\,, \end{aligned}$$

as before.

The explicit diagonalizations for q_1 and q_2 above also lead to various formulas relating the n-ary forms q_i to their $(n-1)$-ary versions. The exploration of these further relations will be left to the reader.

The Forms $q_3,\ q_4$

Coming now to the form q_3, we first note, in the spirit of $(*)$, that $q_3(\mathbf{x})=q_1(\mathbf{x})+q_2(\mathbf{x})$. However, the two diagonalizations obtained for q_1 and q_2 are *not* mutually compatible, so just adding them does not lead to a diagonalization of q_3. Thus, we must deal with the diagonalization problem from scratch.

Fortunately, there is a good way out. Upon noting that $\sum_{i,j} i\,x_ix_j=\sum_{i,j} j\,x_ix_j\,$, we can write $q_3=2f$ for the quadratic form $f(\mathbf{x})=\sum_{i,j} i\,x_ix_j$. Now f factors into $\big(\sum_i i\,x_i\big)\big(\sum_j x_j\big)$. Therefore, using the identity $uv=\left[(u+v)^2-(u-v)^2\right]/4\,$, we get

$$\begin{aligned} q_3(\mathbf{x}) &= \frac{2}{4}\left[\left(\textstyle\sum_i i\,x_i+\sum_i x_i\right)^2-\left(\sum_i i\,x_i-\sum_i x_i\right)^2\right]\\ &= \frac{1}{2}\left[\big(2x_1+3x_2+\cdots+(n+1)x_n\big)^2-\big(x_2+2x_3+\cdots+(n-1)x_n\big)^2\right]. \end{aligned}$$

This yields the diagonalization $q_3=\frac{1}{2}\big(z_1^2-z_2^2\big)$ with $z_1=2x_1+3x_2+\cdots+(n+1)x_n$ and $z_2=x_2+2x_3+\cdots+(n-1)x_n$. In particular, q_3 has rank 2 and signature

0, and we get the interesting bonus conclusion that, for real variables $x_1, \ldots, x_n$, we have $\sum_{i,j} (i+j)\, x_i x_j \geq 0$ iff

$$|2x_1 + 3x_2 + \cdots + (n+1)x_n| \geq |x_2 + 2x_3 + \cdots + (n-1)x_n|\,.$$

We observe in passing that the above method can actually be used to handle the more general form $q := \sum_{i,j} (a_i + b_j)\, x_i x_j$, for arbitrary elements a_i, b_j in the field. One sets $c_i = \big(a_i + b_i\big)/2$ to rewrite q as $2g$ for $g = \sum_{i,j} c_i\, x_i x_j$, after which we can proceed as before.

Finally, we come to the form q_4. The diagonalization of q_4 would have been pretty difficult without the work above. Given what we have already done, however, a solution can be worked out. We start out again by noting that $q_4(\mathbf{x}) = q_2(\mathbf{x}) - q_1(\mathbf{x})$. This in itself does not yield a diagonalization for q_4, but we can further replace $q_2(\mathbf{x})$ by $q_3(\mathbf{x}) - q_1(\mathbf{x})$ to write $q_4(\mathbf{x})$ as $q_3(\mathbf{x}) - 2q_1(\mathbf{x})$. Now introduce the new variables $y_i = x_i + x_{i+1} + \cdots + x_n$ $(1 \leq i \leq n)$. Using our earlier results on q_1 and q_3, we can write

$$\begin{aligned}
q_4(\mathbf{x}) &= 2(x_1 + \cdots + x_n)(x_1 + 2x_2 + \cdots + nx_n) - 2(y_1^2 + \cdots + y_n^2) \\
&= 2y_1(y_1 + y_2 + \cdots + y_n) - 2(y_1^2 + \cdots + y_n^2) \\
&= 2y_1(y_2 + \cdots + y_n) - 2(y_2^2 + \cdots + y_n^2) \\
&= -2 \sum_{i=2}^{n} \left[\left(\frac{y_1}{2} \right)^2 - y_1 y_i + y_i^2 \right] + 2(n-1) \left(\frac{y_1}{2} \right)^2 \\
&= \frac{n-1}{2}\, y_1^2 - \frac{1}{2} \sum_{i=2}^{n} \big(y_1 - 2y_i\big)^2 \\
&= (n-1)\big(x_1 + \cdots + x_n\big)^2/2 - \big(x_1 - x_2 - \cdots - x_n\big)^2/2 \\
&\quad\; - \big(x_1 + x_2 - x_3 - \cdots - x_n\big)^2/2 - \cdots - \big(x_1 + x_2 + \cdots + x_{n-1} - x_n\big)^2/2.
\end{aligned}$$

This gives the desired diagonalization of $q_4(\mathbf{x})$ (over the rationals). In particular, we conclude that, like q_2, q_4 has rank n and signature $2-n$. Note that the above diagonalization of q_4 also leads to a criterion for $q_4(\mathbf{x})$ to be nonnegative for a given real vector $\mathbf{x}$.

As further exercises, the reader may try to diagonalize the following quadratic forms:

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n\,, \quad x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1,$$

and $\sum_{i<j} x_i x_j$.

Epilogue

After I mentioned the above diagonalization of the form $q_4(\mathbf{x})$ in a lecture at the University of Illinois, Professor Kenneth Stolarsky pointed out to me a number of references concerning $q_4(\mathbf{x})$. To my great surprise (as well as delight), it turned out that the form q_4 had been studied by my late Berkeley colleague Raphael M. Robinson sixty five years ago! Robinson [4] posed the computation of the determinant

$$D_n := \det(q_4(\mathbf{x})) = \det(|i-j|)_{1\le i,j\le n}$$

as Problem 3667 in the MAA *Monthly* in 1934, and a little later, posed as Problem 3705 the computation of the signature of $q_4(\mathbf{x})$ over the real numbers. The latter problem was solved by G. Szegö in [6]. Szegö obtained the recurrence relation

$$D_n = -4(D_{n-1} + D_{n-2}),$$

whereby he proved by induction on n that $D_n = (-1)^{n-1}2^{n-2}(n-1)$. Using a formula of Frobenius, Szegö then showed that $q_4(\mathbf{x})$ has real signature $2-n$. Szegö also considered the Hermitian form $\sum_{i,j} |i-j|\, x_i\overline{x}_j$, and related it to the Fourier series expansions of periodic functions. The two solutions Szegö presented to Robinson's problems occupied six pages of the *Monthly*, and read like a paper in itself. This was followed by an even longer "Editorial Note", which occupied another eight pages. The total number of pages, 14, must have been a record for the amount of space ever devoted by the MAA *Monthly* to the solution of a single problem. However, the problem of an explicit diagonalization of $q_4(\mathbf{x})$ *over the rationals* was not considered by Robinson or Szegö.

A few years later, in 1937, I. J. Schoenberg [5: §6] considered the quadratic form

$$\sum_{1\le i,j\le n} |P_i - P_j|^\alpha x_i x_j \quad (0 < \alpha < 2)\,,$$

where $P_1, \ldots, P_n$ are distinct points in an euclidean space, and $|P_i - P_j|$ denote their euclidean distances. Schoenberg showed that this form is nonsingular with real signature $2-n$, and that it is negative definite on the hyperplane $\sum_i x_i = 0$. In the case when $|P_i - P_j| = |i-j|$ and $\alpha = 1$, this, of course, also follows from our diagonalization of the form $q_4(\mathbf{x})$. I am very much indebted to Professor K. Stolarsky for pointing out the above pertinent references.

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