2.1. **Let $G = \dot{F}/\dot{F}^2$, and let $\mathbb{Z}G$ denote the integral group ring of G. Let J be the ideal of $\mathbb{Z}G$ generated by the expressions $a\dot{F}^2 + b\dot{F}^2 - c\dot{F}^2 - d\dot{F}^2$, where $\langle a, b\rangle \simeq \langle c, d\rangle$. Show that there is a natural ring isomorphism between $\mathbb{Z}G/J$ and $\widehat{W}(F)$.**

Solution by Emily Peters, eep@math.berkeley.edu:

We define a map $\phi : \mathbb{Z}G \to \widehat{W}(F)$ by setting $\phi(a\dot{F}^2) := \langle a\rangle$. As this map is defined on the generators of $\mathbb{Z}G$ (namely G itself), we can extend ϕ linearly and multiplicatively to the whole of $\mathbb{Z}G$ and get a ring homomorphism. Now we define $\tilde{\phi} : \mathbb{Z}G/J \to \widehat{W}(F)$ as the composition of the natural map $\tau : \mathbb{Z}G \to \mathbb{Z}G/J$ with ϕ.

To show that $\tilde{\phi}$ is injective, we observe that J is contained in $\ker(\phi)$, because $\phi(a\dot{F}^2+b\dot{F}^2-c\dot{F}^2-d\dot{F}^2) = \langle a\rangle+\langle b\rangle-\langle c\rangle-\langle d\rangle = \langle a, b\rangle-\langle c, d\rangle$, and this is zero in $\widehat{W}(F)$.

To see that $\tilde{\phi}$ is surjective, notice that ϕ is surjective: Any formal difference of two forms $\langle a_1, \ldots, a_m\rangle - \langle b_1, \ldots, b_n\rangle$ can be gotten from the element $a_1\dot{F}^2 + \cdots + a_m\dot{F}^2 - b_1\dot{F}^2 - \cdots - b_n\dot{F}^2$. Then $\tilde{\phi}$ is the composition of two surjective maps, so is itself surjective. This allows us to conclude that $\tilde{\phi}$ is an isomorphism.

2.4. **Show that IF is the unique prime ideal of $W(F)$ that contains the element 2 (= $\langle 1, 1\rangle$). [Do this exercise without looking at VIII.3.4]**

Solution by Emily Peters, eep@math.berkeley.edu:

Suppose that $\mathfrak{p}$ is a prime ideal and $2 \in \mathfrak{p}$. As IF is additively generated by 1-fold Pfister forms, it's enough to show that for all $a \in \dot{F}/\dot{F}^2$, the form $\langle 1, a\rangle$ is in $\mathfrak{p}$. We calculate that $\langle 1, a\rangle \cdot \langle 1, a\rangle = \left\langle 1, a, a, a^2\right\rangle = \langle 1, a, a, 1\rangle = \langle 1, 1\rangle \cdot \langle 1, a\rangle$. As this last product an element of $\mathfrak{p}$ and $\mathfrak{p}$ is prime, we conclude that $\langle 1, a\rangle \in \mathfrak{p}$ for all $a \in \dot{F}/\dot{F}^2$.

2.5. **Prove the following converse to 3.5:** *if $I^2F = 0$, then every binary form over F is universal.*

Solution by Kendra Lockman, kendra(at)math.

Proof. Under the hypotheses, we have for any $a, b \in \dot{F}$,

$$0 = \langle 1, -a \rangle \langle 1, -b \rangle = \langle 1, ab \rangle - \langle a, b \rangle ,$$

so $\langle a, b \rangle = \langle 1, ab \rangle$ in the Witt ring. Note by Witt's cancellation theorem that two binary forms are equal in the Witt ring if and only if they are isometric. Thus $\langle a, b \rangle \cong \langle 1, ab \rangle$, so that every binary form represents 1. We want to show that $\langle a, b \rangle$ is universal, so pick $c \in \dot{F}$. Then $\left\langle ac^{-1}, bc^{-1} \right\rangle$ represents 1, so we can find $x, y \in F$ such that $ac^{-1}x^2 + bc^{-1}y^2 = 1$. Hence $ax^2 + by^2 = c$, as desired. □

2.12. **In working out Example 5.4, we have shown that, if $c \in \dot{F}$ is a sum of four squares in F, then $\langle 1, 1, 1, 1 \rangle \cong \langle c, c, c, c \rangle$. Use this to show that the set of such elements c forms a subgroup of $\dot{F}$.**

Solution by Dave Freeman, `dfreeman@math`.

Solution. Suppose c, d are sums of four squares. Then

$$\langle 1, 1, 1, 1 \rangle \cong \langle c, c, c, c \rangle \cong \langle d, d, d, d \rangle .$$

Multiplying by $\langle d \rangle$ gives

$$\langle cd, cd, cd, cd \rangle \cong \left\langle d^2, d^2, d^2, d^2 \right\rangle \cong \langle 1, 1, 1, 1 \rangle ,$$

so cd is a sum of four squares. □

2.15. **Let F_1, F_2 be subfields of a field such that F_1 is quadratically closed and F_2 is euclidean. Show that $F_1 \cap F_2$ is euclidean.**

Solution by Dave Freeman, `dfreeman@math`.

Solution. Suppose $a \in F_1 \cap F_2$, and a is a square in F_2. Let b be a square root of a in F_2. Since F_1 is quadratically closed, $b \in F_1$. Thus $b \in F_1 \cap F_2$, and a is a square in $F_1 \cap F_2$. Now suppose $-a$ is a square in F_2. By the same reasoning, $-a$ has a square root in F_1, so $-a$ is a square in $F_1 \cap F_2$. Since F_2 is euclidean, it has only two square classes, so these two cases cover all $a \in F_1 \cap F_2$, and we conclude that $F_1 \cap F_2$ has at most two square classes. Since only one of $\{a, -a\}$ is a square in F_2, there are exactly two square classes. □

2.16. **Show that a field F is euclidean if (and only if) $W(F)$ is an infinite cyclic group.**

Solution by Dave Freeman, `dfreeman@math`.

Solution. ($\Rightarrow$): Suppose F is euclidean. Let $q = \langle a_1, \ldots, a_n\rangle \in W(F)$. Since each a_i is a square or minus a square, $q \cong r\langle 1\rangle + s\langle -1\rangle \cong (r - s)\langle 1\rangle$ for some positive integers r, s with $r+s = n$. Thus $W(F) \subset \mathbb{Z}\langle 1\rangle$, and since F is formally real, $\mathbb{Z}\langle 1\rangle \subset W(F)$. Thus $W(F) \cong \mathbb{Z}$.

($\Leftarrow$): Suppose $W(F) \cong \mathbb{Z}$. Then $W(F)$ is generated by $\langle 1\rangle$, so for any $a \in F$, $\langle a\rangle \cong k\langle 1\rangle$ for some $k \in \mathbb{Z}$. By dimension counting, k must be ± 1, so a is a square or minus a square in F. Furthermore, $n\langle 1\rangle$ is anisotropic for all n, so F is formally real and we conclude that F is euclidean. $\square$

2.19. **Let $a \in \dot{F}$, and let q be a form over F with $\dim(q) = 2m$. Show that $q \cong a \cdot q$ iff $q \cong q_1 \perp \cdots \perp q_m$, where each q_i is a binary form such that $q_i \cong a \cdot q_i$.**

Solution by Alex Dugas, asdugas@math.berkeley.edu

We shall prove the "only if" direction by induction on m, noting that the converse is trivial. If $m = 1$ we have nothing to prove. So assume that the result is true for forms of even dimensions smaller than $2m$, and let $\langle a_1, \ldots, a_{2m}\rangle$ be a diagonalization of q. We then have

$$q \cong \langle a_1, \ldots, a_{2m}\rangle \cong \langle aa_1, \ldots, aa_{2m}\rangle .$$

Hence aa_1 is represented by q and we can write $aa_1 = a_1x_1^2+a_2x_2^2+\cdots+ a_{2m}x_{2m}^2$ for certain $x_i \in F$. Thus, $\langle a_2, \ldots, a_{2m}\rangle \cong \left\langle a_2x_2^2 + \cdots + a_{2m}x_{2m}^2, a_3', \ldots, a_{2m}'\right\rangle$ for some $a_i' \in \dot{F}$, and we have $q \cong \left\langle a_1, a_2x_2^2 + \cdots + a_{2m}x_{2m}^2, a_3', \ldots, a_{2m}'\right\rangle$. The point of writing q in this way is that now aa_1 is represented by the binary form $\left\langle a_1, a_2x_2^2 + \cdots + a_{2m}x_{2m}^2\right\rangle$ which is an orthogonal summand of q. Now, since this binary form represents aa_1, comparing determinants we see that

$$\begin{aligned}\left\langle a_1, a_2x_2^2 + \cdots + a_{2m}x_{2m}^2\right\rangle &\cong \left\langle aa_1, a(a_2x_2^2 + \cdots + a_{2m}x_{2m}^2)\right\rangle \\ &\cong a \cdot \left\langle a_1, a_2x_2^2 + \cdots + a_{2m}x_{2m}^2\right\rangle .\end{aligned}$$

Now $q \cong a \cdot q$ implies that we must also have $\langle a'_3, \ldots, a'_{2m}\rangle \cong a \cdot \langle a'_3, \ldots, a'_{2m}\rangle$ by the Witt cancellation theorem. Finally, applying the inductive hypothesis, and setting $q_1 := \left\langle a_1, a_2x_2^2 + \cdots + a_{2m}x_{2m}^2\right\rangle$, we have

$$q \cong q_1 \perp \left\langle a'_3, \ldots, a'_{2m}\right\rangle \cong q_1 \perp q_2 \perp \cdots \perp q_m,$$

where each q_i is a binary form isometric to $a \cdot q_i$.

2.20. **Let q be an even-dimensional form. If $q \cong -q$, show that the form $q \otimes q$ is hyperbolic. (In the language of Witt rings, $2q = 0 \implies q^2 = 0$ in $W(F)$.) (Hint. Use exercise 19 with $a = -1$.)**

Solution by Alex Dugas, asdugas@math.berkeley.edu

Clearly we may assume q is anisotropic. Using the previous exercise, we can write $q \cong q_1 \perp \cdots \perp q_m$ where each q_i is a binary form with $q_i \cong -q_i$. Now

$$q \otimes q \cong \sum_{1 \le i,j \le m} q_i \otimes q_j \cong \sum_{i=1}^{m} q_i \otimes q_i \perp \sum_{1 \le i < j \le m} (q_i \otimes q_j \perp q_j \otimes q_i),$$

where the summations are to be read as orthogonal sums. Thus it suffices to show that $q_i \otimes q_i$ and $q_i \otimes q_j \perp q_j \otimes q_i$ are hyperbolic for all $i \neq j$.

Dealing with the latter first, and using the commutativity of $\otimes$, we have

$$\begin{aligned} q_i \otimes q_j \perp q_j \otimes q_i &\cong q_i \otimes q_j \perp (-q_i) \otimes q_j \\ &\cong (q_i \perp -q_i) \otimes q_j \\ &\cong 2\mathbb{H} \otimes q_j, \end{aligned}$$

which is hyperbolic.

Now let $\langle a_i, b_i\rangle$ be a diagonalization of q_i. Since $q_i \cong -q_i \cong \langle -a_i, -b_i\rangle$, q_i represents $-a_i$. Thus, there are $x, y \in F$ such that $-a_i = a_ix^2 + b_iy^2$, and since $a_i \neq 0$, we obtain $-1 = x^2 + a_ib_i(\frac{y}{a_i})^2$ after dividing both sides by a_i. This shows that -1 is represented by the form $\langle 1, a_ib_i\rangle$, hence, comparing determinants yields $\langle 1, a_i b_i\rangle \cong \langle -1, -a_i b_i\rangle$. We can now compute

$$\begin{aligned} q_i \otimes q_i &\cong \langle a_i, b_i\rangle \otimes \langle a_i, b_i\rangle \\ &\cong \langle a_i, b_i\rangle \otimes \langle -a_i, -b_i\rangle \\ &\cong \langle -1, -a_i b_i, -1, -a_i b_i\rangle \\ &\cong \langle -1, -a_i b_i\rangle \perp \langle 1, a_i b_i\rangle \\ &\cong 2\mathbb{H}. \end{aligned}$$