MATH 110 Lecture Notes 27

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1 Tensor Products

1.1 Construction

Let V and W be vector spaces over a field F. Their tensor product, $V \otimes_F W$, is defined as follows:

Let M be the vector space with basis $\{e_{v\times w}\}_{v\times w \in V\times W}$. That is, $\dim M$ is equal to $|V \times W|$, and every element of M is a finite linear combination of vectors of the form $e_{v\times w}$. Then let N be the subspace spanned all vectors of the following forms:

- $e_{(av_1+v_2)\times w} - ae_{v_1\times w} - e_{v_2\times w}$ where $a \in F$, $v_1, v_2 \in V$, and $w \in W$
- $e_{v\times(aw_1+w_2)} - ae_{v\times w_1} - e_{v\times w_2}$ where $a \in F$, $v \in V$, and $w_1, w_2 \in W$

Then the quotient space M/N is called the tensor product of V and W, denoted $V \otimes_F W$ or $V \otimes W$ when the field F is clear from context. Every element of $V \otimes_F W$ has the form

$$\sum_{i=1}^{n} v_i \otimes w_i$$

for some $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$. The above elements of N give us relations on elements of the tensor product.

1.2 Universal Property

Let $\psi : V \times W \to X$ be a bilinear map over F. That is, $\psi(\cdot, w)$ is a linear map $V \to X$ for each $w \in W$, and $\psi(v, \cdot)$ is a linear map $W \to X$ for each $v \in V$. Then there is a unique linear map $V \otimes_F W \to X$ such that this map composed with the natural bilinear map $(v, w) \mapsto v \otimes w$ is equal to ψ.

Proof. Let $\overline{\psi} : V \otimes W \to X$ be the map $v \otimes w \mapsto \psi(v, w)$. If this is a well-defined linear map, it is clearly the only such with the desired property. In order for it to be well defined, the map $e_{v\times w} \mapsto \psi(v, w)$ must send all the generators of N to 0, which it does by bilinearity.

1.3 Commutativity and Associativity

Theorem. Let V and W be F-vector spaces. There exists a natural isomorphism $V \otimes W \to W \otimes V$ sending $v \otimes w$ to $w \otimes v$.

Proof. We must show that this is a well-defined F-linear map. Then it will be an isomorphism by symmetry, since we can write down a two-sided inverse. Define a linear map $M \to W \otimes V$ sending $e_{v\times w}$ to $w \otimes v$, then check that this map sends the generators of N to 0.

Theorem. Let V, W, and X be F-vector spaces. There exists a natural isomorphism $(V \otimes W) \otimes X \to V \otimes (W \otimes X)$ sending $(v \otimes w) \otimes x$ to $v \otimes (w \otimes x)$.

Proof. Exercise.

1.4 Direct Sums

Let $W = \bigoplus_{i \in I} W_i$. Then there is a natural isomorphism between $\bigoplus_{i \in I} V \otimes W_i$ and $V \otimes W$.

Proof. For each $i \in I$, there is an inclusion map $\varphi_i : W_i \to W$ and a projection map $\psi_i : W \to W_i$ such that $\psi_i \circ \varphi_i$ is the identity on W_i. We can define maps $\bigoplus_{i \in I} V \otimes W_i \to V \otimes W$ by

$$\sum_{i \in I} v_i \otimes w_i \mapsto \sum_{i \in I} v_i \otimes \varphi_i(w_i)$$

and $V \otimes W \to \bigoplus_{i \in I} V \otimes W_i$ by

$$\sum_{i \in I} v_i \otimes w_i \mapsto \sum_{i \in I} v_i \otimes \psi_i(w_i).$$

Then since $\sum_{i \in I} \varphi_i \circ \psi_i$ is the identity on W, these maps are inverses of each other as long as they are both well-defined.

Theorem. Let β_1 be a basis for V_1 and β_2 be a basis for V_2. Then

$$\{v_1 \otimes v_2 \mid v_1 \in \beta_1, v_2 \in \beta_2\}$$

is a basis for $V_1 \otimes V_2$.

Proof. Since $V_2 \cong \bigoplus_{v_2 \in \beta_2} \operatorname{span}\{v_2\}$, there is a natural isomorphism $V_1 \otimes V_2 \cong \bigoplus_{v_2 \in \beta_2} V_1 \otimes \operatorname{span}\{v_2\}$, so we may assume $V_2 = F$. Similarly we may also assume $V_1 = F$. Then we need to show that $\{1 \otimes 1\}$ is a basis for $F \otimes F$. Consider the map

$$a \otimes b \mapsto ab$$

which is linear from $F \otimes F$ to F. This map can be shown to be well-defined and sends $1 \otimes 1$ to 1, so $\{1 \otimes 1\}$ is linearly independent. Then since $a \otimes b = (ab) \otimes 1 = (ab)(1 \otimes 1)$, it is a basis for $F \otimes F$.

1.5 Summary

Any multilinear map $\psi : V_1 \times V_2 \times \cdots \times V_n \to W$ is the composition of a unique linear map $V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W$ and the canonical multilinear map $V_1 \times V_2 \times \cdots \times V_n \to V_1 \otimes V_2 \otimes \cdots \otimes V_n$. The vector space $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is generated by elements of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_n$, and allowing each v_i to range over a basis for V_i gives a basis for $V_1 \otimes V_2 \otimes \cdots \otimes V_n$.

2 Determinants

2.1 Permutations

Let σ be a permutation on $\{1, 2, \ldots, n\}$. Intuitively, σ can be written as a product of transpositions. Let the symbol $(-1)^\sigma$ be equal to 1 if σ is the product of an even number of transpositions and -1 if it is the product of an odd number of transpositions. This is well-defined, since otherwise we could write the identity permutation as a single transposition.

2.2 Characterization of the Determinant

If we think of a square matrix as an ordered list of columns, the determinant is a multilinear map

$$\det : \prod_{i=1}^{n} F^n \to F.$$

Therefore it is the composition of the canonical multilinear map $\prod_{i=1}^n F^n \to \bigotimes_{i=1}^n F^n$ and a unique linear map

$$\varphi : \bigotimes_{i=1}^{n} F^n \to F.$$

Let σ be a permutation on $\{1, 2, \ldots, n\}$. Then

$$\varphi(e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(n)}) = (-1)^\sigma,$$

which follows from case where σ is a transposition and the multiplicativity of the determinant. If $\gamma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a map which is not a permutation, then γ is not one-to-one, so

$$\varphi(e_{\gamma(1)} \otimes e_{\gamma(2)} \otimes \cdots \otimes e_{\gamma(n)}) = 0.$$

These formulas characterizes the linear map φ, and hence the multilinear map det as well.

3 Dual Spaces

For any finite-dimensional vector spaces V and W, there is a natural isomorphism between $V \otimes W^*$ and $\mathcal{L}(W, V)$, given by $v \otimes f \mapsto (w \mapsto f(w) \cdot v)$.