MATH 110 Lecture Notes 26

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1 Positive Definite Operators

Definition. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Then T is said to be *positive definite* if $\langle T(x), x\rangle > 0$ for all nonzero $x \in V$.

Theorem. A self-adjoint operator T is positive definite if and only if all its eigenvalues are positive.

2 Inner Products and Dual Spaces

Let V be a finite-dimensional $\mathbb{R}$-vector space, and let $\varphi : V \to V^*$ be an isomorphism such that $\varphi^t\psi = \varphi$, where $\psi : V \to V^{**}$ is the canonical map.

Theorem. Given a basis β, $[\varphi]_\beta^{\beta^*}$ is symmetric, and φ gives rise to an inner product if and only if this matrix is positive definite with respect to the dot product.

Proof. We have that

$$\begin{aligned}
[\varphi]_\beta^{\beta^*} &= [\varphi^t\psi]_\beta^{\beta^*} \\
&= [\varphi^t]_{\beta^{**}}^{\beta^*}[\psi]_\beta^{\beta^{**}} \\
&= [\varphi^t]_{\beta^{**}}^{\beta^*} \\
&= \left([\varphi]_\beta^{\beta^*}\right)^t.
\end{aligned}$$

For the second claim, observe that

$$\begin{aligned}
\varphi(x)(x) &= \langle [\varphi(x)]_{\beta^*}, [x]_\beta\rangle \\
&= \left\langle [\varphi]_\beta^{\beta^*}[x]_\beta, [x]_\beta \right\rangle.
\end{aligned}$$

The condition that this quantity be positive whenever $x \neq 0$ is the definition of positive definiteness for the matrix $[\varphi]_\beta^{\beta^*}$.