MATH 110 Lecture Notes 25

GSI Carter

August 5, 2008

1 Normal and Self-Adjoint Operators

Definition. Let T be an operator on a finite-dimensional inner product space V. Then T^* is also an operator on V, and T is said to be *normal* if $TT^* = T^*T$. If $T = T^*$, then we call T a *self-adjoint* operator.

Example. Any real, symmetric matrix is self-adjoint. Any real, skew-symmetric matrix is normal but not self-adjoint.

Theorem. Let λ be an eigenvalue of a normal operator T. Then E_λ is both T- and T^*-invariant.

Proof. Any eigenspace of T is always T-invariant. To see that E_λ is also T^*-invariant, let $v \in N(T - \lambda I)$. Then

$$(T - \lambda I)T^*(v) = T^*(T - \lambda I)(v) = T^*(0) = 0,$$

so $T^*(v) \in N(T - \lambda I)$.

Spectral Theorem. Let T be a normal operator on a finite-dimensional complex inner product space V. Then there exists an orthonormal basis for V consisting of eigenvectors of T. If T is self-adjoint as well, then all the eigenvalues of T are real.

Proof. For the first claim, we use induction on $\dim V$. Since every polynomial splits over $\mathbb{C}$, T has an eigenvalue λ. Then E_λ is both T- and T^*-invariant. By exercise 6.4.7, $E_\lambda^\perp$ is also T^*- and T-invariant (since $T^{**} = T$), and $T|_{E_\lambda^\perp}$ is also normal. Since $\dim E_\lambda^\perp < \dim V$, there exists an orthonormal basis for $E_\lambda^\perp$ consisting of eigenvectors of T by induction. Using Gram-Schmidt, we can produce an orthonormal basis for E_λ. Since $V = E_\lambda \oplus E_\lambda^\perp$, the union of these two bases gives a basis for V. Then by the definition of $E_\lambda^\perp$, the resulting basis for V is still orthonormal, which proves the first claim.

Now let λ be an eigenvalue for a self-adjoint operator T. Let v be a nonzero eigenvector in E_λ. Then

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle.$$

Therefore $(\lambda - \overline{\lambda})\langle v, v\rangle = 0$. Since $v \neq 0$, $\langle v, v\rangle \neq 0$, so $\lambda = \overline{\lambda}$. Therefore $\lambda \in \mathbb{R}$.

Corollary. Let v_1 and v_2 be eigenvectors of a normal operator with distinct eigenvalues. Then v_1 and v_2 are orthogonal.

Example. Find an orthonormal basis of eigenvectors for the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

2 Unitary and Orthogonal Operators

Definition. Let T be a linear operator over a finite-dimensional inner product space V, and suppose $\|T(x)\| = \|x\|$ for all $x \in V$. Then T is called *unitary* if V is a complex inner product space and *orthogonal* if V is a real inner product space.

Examples. Any rotation or reflection in $\mathbb{R}^3$ is orthogonal, and the composition of orthogonal operators is orthogonal (same holds for unitary). Also, if $V = C[a,b]$ with the convolution integral inner product and $h \in C[a,b]$ has the property that $|h(x)| = 1$ for all $x \in [a,b]$, then $T(f) = hf$ is either unitary or orthogonal.

Lemma. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for an inner product space V. Then

$$\left\| \sum_{i=1}^n a_i v_i \right\|^2 = \sum_{i=1}^n |a_i|^2.$$

Proof. We have that

$$\begin{aligned}\left\| \sum_{i=1}^n a_i v_i \right\|^2 &= \left\langle \sum_{i=1}^n a_i v_i, \sum_{j=1}^n a_j v_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \langle v_i, v_j \rangle \\ &= \sum_{i=1}^n a_i \overline{a_i} \\ &= \sum_{i=1}^n |a_i|^2.\end{aligned}$$

Theorem. Let T be a linear operator on a finite-dimensional inner product space V. Then the following are equivalent:

1. $TT^* = T^*T = I$.
2. $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$.
3. For any orthonormal basis β for V, $T(\beta)$ is also an orthonormal basis for V.
4. There exists an orthonormal basis β for V such that $T(\beta)$ is an orthonormal basis for V.
5. $\|T(x)\| = \|x\|$ for all $x \in V$.

Proof. It is straightforward to check that (1) implies (2), (2) implies (3), and (3) implies (4). To prove that (5) implies (1), let $x \in V$. Then

$$\langle x, x \rangle = \|x\|^2 = \|T(x)\|^2 = \langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle$$

so that

$$\langle x, (T^*T - I)(x) \rangle = 0.$$

Letting $U = T^*T - I$, U is self-adjoint (since $I^* = I$), so there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ consisting of eigenvectors of U. Then for each i,

$$0 = \langle v_i, U(v_i) \rangle = \overline{\lambda_i} \langle v_i, v_i \rangle = \overline{\lambda_i}.$$

Therefore $U = 0$, which proves the claim.

To prove that (4) implies (5), let $\beta = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V such that $T(\beta)$ is also an orthonormal basis for V. Then for any $x \in V$,

$$x = \sum_{i=1}^n a_i v_i$$

so that $\|x\|^2 = \sum_{i=1}^n |a_i|^2$. Then

$$T(x) = \sum_{i=1}^n a_i T(v_i).$$

Since $T(\beta)$ is also orthonormal, $\|T(x)\|^2 = \sum_{i=1}^n |a_i|^2$.

Definition. The matrices A and B are said to be *unitarilty equivalent* if there is a unitary matrix P such that $A = P^*BP$. (Note that this is stronger than similarity.) Orthogonally equivalent matrices are defined similarly.

Corollary. Any normal operator on a finite-dimensional inner product space V is unitarily or orthogonally equivalent to a diagonal matrix.