MATH 110 Lecture Notes 24

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August 4, 2008

1 Orthogonal Projections

Let W be a finite-dimensional subspace of an inner product space V, and let $\{w_1, \ldots, w_k\}$ be an orthonormal basis for W. Then for any $v \in V$, the orthogonal projection of v onto W is given by

$$\sum_{i=1}^{k} \langle v, w_i \rangle w_i.$$

It is clear that this vector lies in W. To justify the terminology, two additional properties should hold:

- The orthogonal projection of a vector already in W is itself.
- The difference between v and its projection should lie in $W^{\perp}$.

The first property is a previous theorem. To prove the second property, observe that

$$\left\langle v - \sum_{i=1}^{k} \langle v, w_i \rangle w_i, w_j \right\rangle = \langle v, w_j \rangle - \sum_{i=1}^{k} \langle v, w_i \rangle \cdot \langle w_i, w_j \rangle = 0$$

for all j.

Example. Compute the orthogonal projection of $(1, 0, 0)$ onto the plane $x + y + z = 0$.

2 Adjoint Operators

Lemma. Let $\langle \cdot, \cdot \rangle$ be a complex inner product, and let $[\cdot, \cdot]$ be its real part. Then

$$\langle x, y \rangle = [x, y] - i[ix, y]$$

for all x, y.

Proof. We need to show that $-[ix, y]$ is the imaginary part of $\langle x, y \rangle$. The imaginary part of a complex number z is the same as the real part of $\frac{z}{i} = -iz$, so the imaginary part of $\langle x, y \rangle$ is the real part of $-i\langle x, y \rangle = -\langle ix, y \rangle$.

Theorem. Let V_1 and V_2 be finite-dimensional inner product spaces over $\mathbb{R}$ or $\mathbb{C}$ with inner products $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, respectively, and let $T : V_1 \to V_2$ be linear. Then there is a unique linear operator $T^* : V_2 \to V_1$, called the adjoint of T, such that

$$\langle T(x_1), x_2 \rangle_2 = \langle x_1, T^*(x_2) \rangle_1$$

for all $x_1 \in V_1$ and $x_2 \in V_2$.

Proof. First we will prove existence. Suppose V_1 and V_2 are real inner product spaces, and let $\varphi_1 : V_1 \to V_1^*$ be the map $y \mapsto \langle \cdot, y\rangle_1$ and $\varphi_2 : V_2 \to V_2^*$ be the map $y \mapsto \langle \cdot, y\rangle_2$. Then the above equation becomes

$$(\varphi_2(x_2) \circ T)(x_1) = \varphi_2(x_2)(T(x_1)) = \varphi_1(T^*(x_2))(x_1)$$

for all $x_1 \in V_1$ and $x_2 \in V_2$. We can let $T^* = \varphi_1^{-1} \circ T^t \circ \varphi_2$. Then

$$(\varphi_1 \circ T^*)(x_2)(x_1) = T^t(\varphi_2(x_2))(x_1) = \varphi_2(x_2)(T(x_1))$$

for all $x_1 \in V_1$ and $x_2 \in V_2$ as desired.

Now let V_1 and V_2 be complex inner product spaces. Then if $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$ are the real parts of $\langle \cdot,\cdot\rangle_1$ and $\langle \cdot,\cdot,\rangle_2$, respectively, V_1 and V_2 considered as $\mathbb{R}$-vector spaces are real inner product spaces, and T is also $\mathbb{R}$-linear. Then there is an $\mathbb{R}$-linear map $T^* : V_2 \to V_1$ such that

$$[T(x_1), x_2]_2 = [x_1, T^*(x_2)]_1$$

for all $x_1 \in V_1$ and $x_2 \in V_2$. By the lemma,

$$\begin{aligned}\langle T(x_1), x_2\rangle_2 &= [T(x_1), x_2]_2 - i[iT(x_1), x_2]_2\\ &= [T(x_1), x_2]_2 - i[T(ix_1), x_2]_2\\ &= [x_1, T^*(x_2)]_1 - i[ix_1, T^*(x_2)]_1\\ &= \langle x_1, T^*(x_2)\rangle_1\end{aligned}$$

for all $x_1 \in V_1$ and $x_2 \in V_2$. Now we must show that $T^*(iy) = iT^*(y)$ for all $y \in V_2$, so that T^* is $\mathbb{C}$-linear and not merely $\mathbb{R}$-linear. For any $x \in V_1$,

$$\begin{aligned}\langle x, T^*(iy) - iT^*(y)\rangle_1 &= \langle x, T^*(iy)\rangle_1 + i\langle x, T^*(y)\rangle_1\\ &= \langle T(x), iy\rangle_2 + i\langle T(x), y\rangle_2\\ &= -i\langle T(x), y\rangle_2 + i\langle T(x), y\rangle_2\\ &= 0,\end{aligned}$$

so $T^*(iy) - iT^*(y) = 0$.

Suppose T_1 and T_2 were two adjoint operators to T. Let $y \in V_2$. Then for all $x \in V_1$,

$$\langle x, T_1(y) - T_2(y)\rangle_1 = \langle x, T_1(x)\rangle_1 - \langle x, T_2(y)\rangle_1 = \langle T(x), y\rangle_2 - \langle T(x), y\rangle_2 = 0.$$

Therefore $T_1(y) = T_2(y)$ for all $y \in V_2$, so the adjoint of a linear transformation is unique.

Theorem. Let β_1 and β_2 be orthonormal bases for finite-dimensional inner product spaces V_1 and V_2, respectively, and let $T : V_1 \to V_2$ be linear. Then $[T^*]_{\beta_2}^{\beta_1} = \left([T]_{\beta_1}^{\beta_2}\right)^*$.

Proof. Let $A = [T]_{\beta_1}^{\beta_2}$, and let $B = [T^*]_{\beta_2}^{\beta_1}$. Then if $\beta_1 = \{v_1, \ldots, v_m\}$ and $\beta_2 = \{w_1, \ldots, w_n\}$, B_{ij} is the coefficient in front of v_i for $T^*(w_j)$. Since β_1 and β_2 are orthonormal,

$$B_{ij} = \langle T^*(w_j), v_i\rangle = \overline{\langle v_i, T^*(w_j)\rangle} = \overline{\langle T(v_i), w_j\rangle} = \overline{A_{ji}} = A^*_{ij}.$$

Therefore $B = A^*$.

Exercise 6.3.18. Let A be an $n \times n$ matrix. Prove that $\det(A^*) = \overline{\det A}$.