MATH 110 Lecture Notes 23

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1 Inner Products

Exercise 6.1.27. Let $\| \cdot \|$ be a norm on a real vector space V satisfying the parallelogram law. That is, for all $x, y \in V$,

$$\|x+y\| + \|x-y\| = 2\|x\|^2 + 2\|y\|^2.$$

Define

$$\langle x, y\rangle = \frac{1}{4}\left[\|x+y\|^2 - \|x-y\|^2\right].$$

First observe that $\langle x, x\rangle = \|x\|^2$ and $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$. We only need to show that this function is linear in the first variable.

(a) Prove $\langle x, 2y\rangle = 2\langle x, y\rangle$ for all $x, y \in V$. (We did this yesterday.)

(b) Prove $\langle x+u, y\rangle = \langle x, y\rangle + \langle u, y\rangle$ for all $x, u, y \in V$. Consider

$$\begin{aligned}
\langle x+u, y\rangle - \langle x, y\rangle &= \frac{1}{4}\left[\|x+u+y\|^2 - \|x+u-y\|^2 - \|x+y\|^2 + \|x-y\|^2\right] \\
&= \frac{1}{4}\left[\left\|\left(x+\frac{u}{2}\right)+\left(y+\frac{u}{2}\right)\right\|^2 + \left\|\left(x+\frac{u}{2}\right)-\left(y+\frac{u}{2}\right)\right\|^2\right] \\
&\quad -\frac{1}{4}\left[\left\|\left(x+\frac{u}{2}\right)+\left(-y+\frac{u}{2}\right)\right\|^2 + \left\|\left(x+\frac{u}{2}\right)-\left(-y+\frac{u}{2}\right)\right\|^2\right] \\
&= \frac{1}{4}\left[2\left\|x+\frac{u}{2}\right\|^2 + 2\left\|y+\frac{u}{2}\right\|^2 - 2\left\|x+\frac{u}{2}\right\|^2 - 2\left\|-y+\frac{u}{2}\right\|^2\right] \\
&= \frac{1}{2}\left[\left\|y+\frac{u}{2}\right\|^2 - \left\|y-\frac{u}{2}\right\|^2\right] \\
&= 2\langle y, u/2\rangle \\
&= \langle y, u\rangle \\
&= \langle u, y\rangle.
\end{aligned}$$

(c) Prove $\langle nx, y\rangle = n\langle x, y\rangle$ for every positive integer n and every $x, y \in V$. (This follows from part (b) by induction on n.)

(d) Prove $m\langle \frac{1}{m}x, y\rangle = \langle x, y\rangle$ for every positive integer m and every $x, y \in V$. (By part (c), $\langle m(\frac{1}{m}x), y\rangle = m\langle \frac{1}{m}x, y\rangle$.)

(e) Prove $\langle rx, y\rangle = r\langle x, y\rangle$ for every rational number r and every $x, y \in V$. (If $r > 0$, we can write $r = \frac{m}{n}$, where m and n are both positive integers. Then

$$\langle rx, y\rangle = m\langle \frac{x}{n}, y\rangle = \frac{m}{n}\left[n\langle \frac{1}{n}x, y\rangle\right] = \frac{m}{n}\langle x, y\rangle = r\langle x, y\rangle.$$

If $r = 0$, one only needs to check that $\langle 0, y\rangle = 0$ directly from the above formula. If $r < 0$, then $-r > 0$, so $\langle -rx, y\rangle = -r\langle x, y\rangle$. Then

$$0 = \langle 0, y\rangle = \langle rx - rx, y\rangle = \langle rx, y\rangle + \langle -rx, y\rangle = \langle rx, y\rangle - r\langle x, y\rangle$$

which completes the proof.)

(f) Prove that $|\langle x, y\rangle| \leq \|x\|\|y\|$ for every $x, y \in V$. (Use the above formula and the inequalities $\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ and $\|x - y\|^2 \geq \|x\|^2 - 2\|x\|\|y\| + \|y\|^2$.)

(g) Prove that for every $c \in \mathbb{R}$, every rational number r, and every $x, y \in V$,

$$|c\langle x, y\rangle - \langle cx, y\rangle| = |(c - r)\langle x, y\rangle - \langle (c - r)x, y\rangle| \leq 2|c - r|\|x\|\|y\|.$$

(The first equality follows from part (e). Then use the triangle inequality and apply part (f) to both $|(c - r)\langle x, y\rangle|$ and $|\langle (c - r)x, y\rangle|$.)

(h) Given $c \in \mathbb{R}$ and $x, y \in V$, we have that

$$|c\langle x, y\rangle - \langle cx, y\rangle| \leq 2|c - r|\|x\|\|y\|$$

for all $r \in \mathbb{Q}$. Therefore $c\langle x, y\rangle = \langle cx, y\rangle$. Therefore $\langle \cdot, \cdot\rangle$ is linear in the first variable. The other two inner product axioms follow in a straightforward way from the above formula.

2 Orthonormal Bases

Definition. A subset S of an inner product space V is called *orthogonal* if $\langle x, y\rangle = 0$ for any distinct $x, y \in S$. If, additionally, each $\|x\| = 1$ for all $x \in S$, then S is called *orthonormal.*

Theorem. Let $\{v_1, \ldots, v_k\}$ be an orthonormal basis for an inner product space V. Then for any $y \in V$,

$$y = \sum_{i=1}^{k} \langle y, v_i\rangle v_i.$$

Proof. We can write

$$y = \sum_{i=1}^{k} a_i v_i.$$

Then for each j,

$$\langle y, v_j\rangle = \sum_{i=1}^{k} a_i \langle v_i, v_j\rangle = a_j.$$

Gram-Schmidt. Let $\{w_1, \ldots, w_k\}$ be a basis for a vector space V. If

$$v_j = w_j - \sum_{i=1}^{j-1} \frac{\langle w_j, v_i\rangle}{\|v_i\|^2} \cdot v_i$$

then $\{v_1, \ldots, v_k\}$ is an orthogonal basis for V.

Proof. We need to check by induction on j that $\langle v_j, v_i\rangle = 0$ whenever $i < j$:

$$\langle v_j, v_i\rangle = \langle w_j, v_i\rangle - \sum_{m=1}^{j-1} \frac{\langle w_j, v_m\rangle}{\|v_m\|^2} \cdot \langle v_m, v_i\rangle = \langle w_j, v_i\rangle - \langle w_j, v_i\rangle = 0.$$

Then we must also check that $\operatorname{span}\{w_1, \ldots, w_j\} = \operatorname{span}\{v_1, \ldots, v_j\}$ by induction on j. For the base step, there is nothing to prove, and for the induction step, it suffices to show that $v_j \in \operatorname{span}\{w_1, \ldots, w_{j-1}\}$ and $w_j \in \operatorname{span}\{v_1, \ldots, v_{j-1}\}$.

Definition. Let W be a subspace of an inner product space V. Then the *orthogonal complement* of W, $W^\perp$, is the set

$$W^\perp = \{v \in V \mid \langle v, w\rangle = 0 \forall w \in W\}.$$

Because inner products are linear in the first variable, $W^\perp$ is also a subspace of V.

Exercise 6.2.6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y\rangle \neq 0$.

Proof. Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for W, and let $w = \sum_{i=1}^{n} \langle x, w_i\rangle w_i$. Then $x - w \in W^\perp$, since for each j,

$$\langle w, w_j\rangle = \sum_{i=1}^{n} \langle x, w_i\rangle \cdot \langle w_i, w_j\rangle = \langle x, w_j\rangle.$$

Then

$$\langle x, x - w\rangle = \langle x - w, x - w\rangle + \langle w, x - w\rangle = \|x - w\|^2 > 0$$

since $x \notin W$.