MATH 110 Lecture Notes 22

GSI Carter

July 30, 2008

1 Inner Product Spaces

1.1 Definition

Let V be a $\mathbb{C}$-vector space. Then an *inner product* on V is any function from ordered pairs in V to $\mathbb{C}$, $\langle \, , \, \rangle$, with the following properties.

- Linear in the first variable. That is, for any $\alpha \in \mathbb{C}$ and any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

  $$\langle \alpha\mathbf{u}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle$$

  and

  $$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle.$$

- For any $\mathbf{u}, \mathbf{v} \in V$,

  $$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}.$$

- For any nonzero $\mathbf{u} \in V$, $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ (in particular, the inner product of any nonzero vector with itself is a real number).

As a consequence of these three rules, any inner product will be *conjugate linear* in the second variable. That is,

$$\langle \mathbf{u}, \alpha\mathbf{w} \rangle = \overline{\langle \alpha\mathbf{w}, \mathbf{u} \rangle} = \overline{\alpha \langle \mathbf{w}, \mathbf{u} \rangle} = \overline{\alpha} \langle \mathbf{u}, \mathbf{w} \rangle$$

and

$$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle.$$

We can also have an inner product on an $\mathbb{R}$-vector space. In that case, the inner product of any two vectors should be a real number, and all above references to scalars apply only to real numbers.

A vector space with an inner product is called an *inner product space*. In any inner product space, we can define a notion of the length of a vector, or it's norm:

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

The last axiom in the definition of an inner product space guarantees that all vectors have non-negative, real length, and that only the zero vector has length zero.

1.2 Dot Product

The simplest example of an inner product is the dot product on $\mathbb{C}^n$. Given $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, the dot product of $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{n} a_i \overline{b_i}.$$

The length of $\mathbf{e}_i$ is 1.

Example. $\|(1, -2, 0, 2)\| = \sqrt{1 + 4 + 4} = 3.$

1.3 Convolution Integral

Another important example of an inner product is the convolution integral inner product on $C[a, b]$. This is defined as

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}\, dx$$

for all $f, g \in C[a, b]$. It is clear that this satisfies the first two axioms. For the third, let $f \in C[a, b]$ be any nonzero continuous function. Then there exists $x_0 \in (a, b)$ such that $f(x_0) \neq 0$. Let $\epsilon = \frac{|f(x_0)|}{2}$. Then there exist $\delta > 0$ such that

- $a < x_0 - \delta < x_0 + \delta < b$; and
- $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Then for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$|f(x)| > |f(x_0)| - \epsilon = \frac{|f(x_0)|}{2}$$

so that

$$\begin{aligned}
\langle f, f \rangle &= \int_a^b f(x)\overline{f(x)}\, dx \\
&= \int_a^b |f(x)|^2\, dx \\
&\geq \int_{x_0-\delta}^{x_0+\delta} |f(x)|^2\, dx \\
&\geq \int_{x_0-\delta}^{x_0+\delta} \frac{|f(x_0)|^2}{4}\, dx \\
&= \frac{\delta |f(x_0)|^2}{2} \\
&> 0.
\end{aligned}$$

Then since it is clear that $\langle 0, 0 \rangle = 0$, the third axiom is satisfied.

Exercise 6.1.11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

2 Inner Products and Dual Spaces

Let V be a finite-dimensional vector space over $\mathbb{R}$. Then an inner product on V is linear in both variables. For each $x \in V$, the map

$$y \mapsto \langle y, x\rangle$$

is a linear functional on V. Since a real inner product is also linear in the second variable, the function

$$x \mapsto \langle \cdot, x\rangle$$

is a linear transformation from V to V^*. Call this linear transformation φ. We can use the properties of inner products to deduce things about φ.

Theorem. The map φ is an isomorphism.

Proof. Since $\dim V = \dim V^*$, it is enough to show that φ is one-to-one. If $\langle \cdot, x\rangle$ is the zero map from V to $\mathbb{R}$, then $x = 0$, since otherwise $\langle x, x\rangle > 0$.

Theorem. Let $\psi : V \to V^{**}$ be the canonical isomorphism. Then $\varphi^t\psi = \varphi$.

Proof. Let $x \in V$. Then

$$\varphi^t\psi(x) = \varphi^t(\hat{x}) = \hat{x}\varphi = \varphi(x).$$

Theorem. Let $\varphi : V \to V^*$ be an isomorphism such that $\varphi(x)(x) > 0$ for all $x \in V$ and $\varphi^t\psi = \varphi$, with ψ as above. Then

$$\langle y, x\rangle = \varphi(x)(y)$$

is an inner product on V.

Proof. Since $\varphi(x) \in V^*$ is linear, we get linearity in the first variable. Then since $\varphi^t\psi = \varphi$,

$$\begin{aligned}\langle x, y\rangle &= \varphi(y)(x) \\ &= \left[\varphi^t\psi(y)\right](x) \\ &= \left[\hat{y}\varphi\right](x) \\ &= \varphi(x)(y) \\ &= \langle y, x\rangle.\end{aligned}$$

The last inner product axiom is given.

3 Norms

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Then a *norm* $\|\cdot\|$ is a function from V to $\mathbb{R}$ such that:

- For all $x \in V$, $\|x\| \geq 0$, and $x = 0$ if $\|x\| = 0$.
- For all scalars a and all $x \in V$, $\|ax\| = |a| \cdot \|x\|$.
- For all $x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.

Note that any norm as defined from an inner product above is also a norm by this definition.

Exercise 6.1.24. Prove that the following are norms on the given vector spaces V.

(a) $V = M_{n\times n}(F)$, $\|A\| = \max_{i,j} |A_{ij}|$

(b) $V = C([0,1])$, $\|f\| = \max_{t\in[0,1]} |f(t)|$

(c) $V = C([0,1])$, $\|f\| = \int_0^1 |f(t)|\, dt$

(d) $V = \mathbb{R}^2$, $\|(a,b)\| = \max\{|a|, |b|\}$