MATH 110 Lecture Notes 20

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July 28, 2008

1 Midterm 2 Summary

- Systems of equations and rank
  - When is a system of equations consistent or inconsistent?
  - rank of a matrix: definition and properties
- Determinants
  - definition
  - be able to compute them
  - properties
- Eigenvalues
  - how to find eigenvalues and eigenvectors
  - Cayley-Hamilton theorem (statement and applications, including what it means to plug an operator into a polynomial, not the proof of the theorem)
- Jordan canonical form
  - definition
  - the T-invariant decomposition used to put a matrix in Jordan form
  - be able to compute it for a given matrix
  - relation of minimal polynomial to Jordan form

2 Dual Spaces

Let V be a vector space over a field F. Then a *linear functional* on V is a linear transformation from V to F. The set of all linear functionals on V forms a vector space over F which we call the *dual space* of V, denoted V^*.

Example. Let $V = F^n$. Then a linear function on V is simply an n-element row matrix. In this example, we see that $\dim V^* = n = \dim V$.

Example. Let V be the set of $n \times n$ matrices over F. Then the trace function is a linear functional on V.

Theorem. Let V be finite dimensional. Then $\dim V = \dim V^*$. (This is a special case of a previous theorem on the dimension of the space of linear transformations.)

Theorem. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for a vector space V over F. For each i, let $f_i : V \to F$ be the linear functional such that

$$f_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then $\beta^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for V^*, which we call the dual basis to β.

Proof. Let $f \in V^*$. Then

$$f = \sum_{i=1}^{n} f(x_i) f_i,$$

which we can see by applying each side to each element of β. Therefore β^* generates V^*. Then since $|\beta^*| = n = \dim V^*$, β^* is a basis for V^*.

Theorem. Let V and W be vector spaces with bases β and γ, respectively. Then for any linear tranformation $T : V \to W$, we define the dual of T, $T^t : W^* \to V^*$, by $T^t(g) = gT$ for all $g \in W^*$. Then $[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$.

Proof. Checking that T^* as defined is a linear transformation from W^* to V^* is straightforward. Let $\beta = \{x_1, \ldots, x_n\}$ and $\gamma = \{y_1, \ldots, y_m\}$, and let $\beta^* = \{f_1, \ldots, f_n\}$ and $\gamma^* = \{g_1, \ldots, g_m\}$ as above. Then we have

$$T^t(g_j) = g_j T = \sum_{i=1}^{n} (g_j T)(x_i) f_i$$

so that the (i, j)-th entry of $[T^t]_{\gamma^*}^{\beta^*}$ is given by $(g_j T)(x_i)$. Since $g_j(T(x_i))$ is the coefficient in front of y_j used to obtain $T(x_i)$, this is also the (j, i)-th entry of $[T]_\beta^\gamma$, which proves the claim.

Theorem. Let V be a vector space over F. For each $x \in V$, there is a linear map $\hat{x} \in V^{**}$ given by $\hat{x}(f) = f(x)$ for each $f \in V^*$. Then there is a one-to-one linear map $V \to V^{**}$ given by

$$x \mapsto \hat{x}.$$

Proof. Proving linearity is straightforward. To prove it is one-to-one, let x be nonzero. Then we can expand $\{x\}$ to a basis β for V, and there exists $f \in V^*$ such that $f(x) = 1$ and $f(y) = 0$ for each $y \in \beta$ other than x. Then $\hat{x}(f) = f(x) = 1$, so $\hat{x} \neq 0$.

Corollary. If V is finite-dimensional, the above map $V \to V^{**}$ is an isomorphism.

Proof. Since the map is one-to-one and $\dim V^{**} = \dim V^* = \dim V$, it is an isomorphism.

Theorem. Let $T : V \to W$ be an isomorphism. Then so is $T^t : W^* \to V^*$.

Proof. If $g \in N(T^t)$, then $gT = 0$, so that

$$g = (gT)T^{-1} = 0 \cdot T^{-1} = 0.$$

If $f \in V^*$, then $fT^{-1} \in W^*$, and

$$T^t(fT^{-1}) = (fT^{-1})T = f.$$

Theorem. Let $U : V \to W$ and $T : W \to Z$ be linear. Then

$$(TU)^t = U^t T^t.$$

Proof. Both sides of the equation are linear transformations from Z^* to V^*. Let $g \in Z^*$. Then

$$(TU)^t(g) = g(TU)$$

and

$$U^t(T^t(g)) = U^t(gT) = (gT)U.$$

Since composition of functions is associative, this completes the proof.

Lemma. Let $T : V \to W$ be linear. Then T is one-to-one if and only if it has a right inverse, and T is onto if and only if it has a left inverse.

Proof. If T is one-to-one, then the induced map $V \to R(T)$ is an isomorphism, and hence has an inverse. If we extend a basis for $R(T)$ to one for W, then the map which is the inverse of $V \to R(T)$ on $R(T)$ and which sends all other basis elements to 0 is a right inverse of T. Conversely, if T has a right inverse U, then

$$N(T) \subseteq N(UT) = \{0\},$$

so T is one-to-one.

If T is onto, then the induced map $V/N(T) \to W$ is an isomorphism, and hence has an inverse. If we choose a basis for $V/N(T)$, there is a linear map $U : V/N(T) \to V$ sending each element $v + N(T)$ in the basis to its chosen representative v. Then since U composed with the canonical map $V \to V/N(T)$ is the identity, the composition of the inverse of $V/N(T) \to W$ with U is a left inverse of T. (Draw a picture of all these maps.)

Conversely, if $U : W \to V$ is a left inverse of T, then

$$R(T) \supseteq R(TU) = W,$$

so T is onto.

Exercise 2.6.20. Let $T : V \to W$ be linear. Then prove that T is one-to-one if and only if T^t is onto, and that T is onto if and only if T^t is one-to-one.

Theorem. Let W be a subspace of a vector space V over F. Let $i : W \to V$ be the inclusion map, let $p : V \to V/W$ be the canonical projection, and let

$$Z = \{f \in V^* \mid f|_W = 0\}.$$

Then $i^t : V^* \to W^*$ is onto and gives an isomorphism between W^* and V^*/Z, and $p^t : (V/W)^* \to V^*$ is one-to-one with image equal to Z (and thus gives an isomorphism between $(V/W)^*$ and Z).

Proof. By exercise 2.6.20, i^t is onto and p^t is one-to-one. We must now show that both $N(i^t)$ and $R(p^t)$ are equal to Z.

Given a map $f \in V^*$, the following are equivalent:

- $f \in N(i^t)$
- $fi = 0$
- $f|_{R(i)} = 0$
- $f \in Z$

This shows that $N(i^t) = Z$.

If $f \in R(p^t)$, there exists a map $g : V/W \to F$ such that $f = gp$. Then since $N(p) = W$, $N(f) \supseteq N(p) = W$, so $f \in Z$. Conversely, if $f \in Z$, $W \subseteq N(f)$, so the map $\overline{f} : V/W \to F$ given by $\overline{f}(v + W) = f(v)$ is well-defined, and $f = \overline{f}p$, so $f \in R(p^t)$.