MATH 110 Lecture Notes 19

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1 Ideals

An *ideal* I in $P(F)$ is a subspace of $P(F)$ with the additional requirement that I is closed under multiplication. That is, if $p(t) \in P(F)$ and $q(t) \in I$, then $p(t)q(t) \in I$.

Examples. The following are all ideals in $P(F)$. We can check each is an ideal by checking closure under addition and multiplication (scalar multiplication is a special case of multiplication, and each of these examples is obviously nonempty).

- $\{0\}$
- $P(F)$
- The set of multiples of a polynomial $p(t)$.
- The ideal generated by two polynomials $p(t)$ and $q(t)$:

$$\{s_1(t)p(t) + s_2(t)q(t) \mid s_1(t), s_2(t) \in P(F)\}$$

- Given a linear operator T on a finite dimensional vector space V, the set

$$\{f(t) \in P(F) \mid f(T) = 0\}$$

  is an ideal.

Theorem. Given a nonzero ideal I in $P(F)$, there is a monic polynomial $p(t)$ such that

$$I = \{q(t)p(t) \mid q(t) \in P(F)\}.$$

Proof. Since I contains at least one nonzero element, we can let $p(t)$ be a nonzero element of I of minimum degree. We can show that every element of I is a multiple of $p(t)$. Let $f(t) \in I$. Then there exist $q(t), r(t) \in P(F)$ such that

$$f(t) = q(t)p(t) + r(t)$$

and $\deg r(t) < \deg p(t)$. Since $p(t) \in I$, so it $q(t)p(t)$, and therefore so is

$$r(t) = f(t) - q(t)p(t).$$

By the choice of $p(t)$, $r(t) = 0$, so $f(t)$ is a multiple of $p(t)$. Since $p(t)$ is nonzero, it has a nonzero leading coefficient a. Then $\frac{p(t)}{a}$ is the desired monic generator of I.

Corollary. The polynomial $p(t)$ as above is unique.

Proof. Let $p_1(t)$ and $p_2(t)$ be two such polynomials. Then they are both monic, and both multiples of each other. Therefore they are equal.

2 The Minimal Polynomial

Definition. Given an operator T on a finite-dimensional vector space V, there is a unique monic generator $p(t)$ of the ideal $I = \{f(t) \in P(F) \mid f(T) = 0\}$ (we know this ideal is nonzero by Cayley-Hamilton). This polynomial is called the *minimal polynomial* of T.

Corollary. The minimal polynomial of an operator divides its characteristic polynomial.

Notation. For an operator T, let $I(T)$ denote the set of polynomials $f(t)$ such that $f(T) = 0$.

Theorem. Let T be an operator on a finite-dimensional vector space V, and let

$$V = \bigoplus_{i=1}^{k} W_i$$

be a T-invariant decomposition of V (that is, each W_i is T-invariant). Then the minimal polynomial of T is equal to the least common multiple of the minimal polynomials of the operators $T|_{W_i}$.

Proof. For any $f(t) \in P(F)$, $f(T)|_{W_i} = f(T|_{W_i})$. Since $f(T) = 0$ if and only if $f(T)|_{W_i} = 0$ for all i, we have that

$$I(T) = \bigcap_{i=1}^{k} I(T|_{W_i}).$$

The ideal on the left is generated by the minimal polynomial of T. The generator of the ideal on the right is a polynomial $f(t)$ having the properties:

- for each i, the minimal polynomial of $T|_{W_i}$ divides $f(t)$
- if the minimal polynomial of $T|_{W_i}$ divides $g(t)$ for each i, then $f(t)$ also divides $g(t)$

The polynomial with these properties is the least common multiple.

Claim. Let B be an $n \times n$ Jordan block with eigenvalue λ. Then the minimal polynomial of B is $(t - \lambda)$.

Proof. Simply check that $(B - \lambda I)^n = 0$ but $(B - \lambda I)^{n-1} \neq 0$.

Corollary. An operator T on a finite-dimensional vector space is diagonalizable if and only if its minimal polynomial is the product of distinct linear factors.

Corollary. Every linear factor of the characteristic polynomial is also a factor of the minimal polynomial.

Examples. From lecture notes 19 from last summer.

Definition. Let T be a linear operator on a finite-dimensional vector space V, and let x be a nonzero vector in V. The polynomial $p(t)$ is called a T-annihilator of x if $p(t)$ is a monic polynomial of least degree for which $p(T)(x) = 0$.

Exercise 7.3.15(a) Prove that the vector x has a unique T-annihilator. Hint: the set $\{p(t) \in P(F) \mid p(T)(x) = 0\}$ is a nonzero ideal.

(b) The T-annihilator of x divides any polynomial $g(t)$ for which $g(T) = T_0$.

(c) If $p(t)$ is the T-annihilator of x and W is the T-cyclic subspace generated by x, then $p(t)$ is the minimal polynomial of $T|_W$, and $\dim W$ equals the degree of $p(t)$. Hint: show the equality of two ideals, and use relations among the vectors $T^m x$ to produce elements of those ideals.

(d) The degree of the T-annihilator of x is 1 if and only if x is an eigenvector of T.