MATH 110 Lecture Notes 18

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1 The Jordan Canonical Form

1.1 Nilpotent Matrices

Here we assume that T is a nilpotent operator on an n-dimensional vector space V with characteristic polynomial $f(t) = (-t)^n$. (This was the case reduced to last time, up to addition of a matrix λI for some scalar λ.) We have a chain of subspaces

$$V \supset R(T) \supseteq R(T^2) \supseteq \cdots \supseteq R(T^{n-1}) \supseteq R(T^n) = \{0\}.$$

This gives us another chain

$$N(T) \supseteq N(T) \cap R(T) \supseteq N(T) \cap R(T^2) \supseteq \cdots \supseteq N(T) \cap R(T^n) = \{0\}.$$

We can construct an ordered basis $v_1, \ldots, v_k$ of $N(T)$ by starting with a basis for $N(T) \cap R(T^{n-1})$, expanding it to a basis for $N(T) \cap R(T^{n-2})$, expanding that to a basis for $N(T) \cap R(T^{n-3})$, and so on. Then for each i there exist cycles

$$v_i = T^{r_i} x_i, T^{r_i - 1} x_i, \ldots, T x_i, x_i$$

whose lengths decrease (not strictly) as i increases. Let $W_i = \operatorname{span}\{T^{r_i} x_i, \ldots, T x_i, x_i\}$. By a previous prelim problem, this cycle is a basis for W_i.

Theorem. With V and W_i as above,

$$V = \sum_{i=1}^{k} W_i.$$

Proof. There is another chain of subspaces

$$N(T) \subseteq N(T^2) \subseteq \cdots \subseteq N(T^n) = V.$$

We will show that $N(T^m) \subseteq \sum_{i=1}^{k} W_i$ by induction on m.

Base step. For $m = 1$, observe that $v_i \in W_i$ for each i.

Induction step. Here $m > 1$. Suppose $v \in N(T^m)$ but $v \notin N(T^{m-1})$. Then there is a cycle

$$T^{m-1} v, T^{m-2} v, \ldots, T v, v$$

with $T^{m-1} v \in N(T)$, so that $T^{m-1} v \in N(T) \cap R(T^{m-1})$. Let j be such that $v_1, \ldots, v_j$ is a basis for $N(T) \cap R(T^{m-1})$. Then there exist $w_1, \ldots, w_j$ such that for each i, $T^{m-1} w_i = v_i$ and $w_i \in W_i$; and there exist scalars such that

$$T^{m-1} v = \sum_{i=1}^{j} a_i v_i = \sum_{i=1}^{j} a_i T^{m-1} w_i.$$

Then $v - \sum_{i=1}^{j} a_i w_i \in N(T^{m-1})$, which proves the claim by induction.

Theorem. With V and W_i as above,

$$V = \bigoplus_{i=1}^{k} W_i.$$

Proof. Suppose $v \in W_j \cap \sum_{i=1, i\neq j}^{k} W_i$. Since each W_i is T-invariant, if $v \neq 0$ we can repeatedly apply T to v to see that

$$v_j \in \sum_{i=1, i\neq j}^{k} W_i$$

which is a contradiction (look at what coefficients could possibly go in front of different terms in the cycles).

Theorem. Let T and V be as above. Then for some basis β, $[T]_\beta$ is block diagonal, with each block having the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(that is, each entry above the main diagonal is 1, while all others are zero).

Proof. Choose the above bases for each W_i.

1.2 Matrices with a Single Eigenvalue

Let T be an operator on a vector space V with characteristic polynomial $f(t) = (\lambda - t)^n$. Then $T - \lambda I$ is nilpotent, so the above applies to it. Then, with respect to the same basis, $T = (T - \lambda I) + \lambda I$ is block diagonal, where each block has the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

1.3 General Matrices

Let T be an operator on a vector space V whose characteristic polynomial splits. Then for each eigenvalue λ with multiplicity m, the above applies to $T|_{N((T-\lambda I)^m)}$, which has only the eigenvalue λ. Then the T-invariant decomposition from last time gives us the Jordan canonical form of T.

2 Examples

See lecture notes 19 from last summer.

3 Problems

Exercise 7.2.6. Let A be a matrix whose characteristic polynomial splits. Prove that A and A^t have the same Jordan canonical form, and are therefore similar. Hint: start with a single nilpotent Jordan block.