MATH 110 Lecture Notes 17

GSI Carter

July 21, 2008

1 Problems

Exercise 5.4.24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. Hint: Use the result of exercise 23, which states that if the sum of eigenvectors with distinct eigenvalues lies in a T-invariant subspace, then so does each individual eigenvector. Hint: It is enough to show that the T-invariant subspace is spanned by eigenvectors of T which it contains, so choose a basis for the subspace and use the fact that T is diagonalizable.

Exercise 5.4.25(a). Let T and U be diagonalizable operators on a finite-dimensional vector space V. Prove that if $UT = TU$, then U and T are simultaneously diagonalizable. Hint: For any eigenvalue λ of T, show that E_λ is U-invariant, annd apply exercise 24 to obtain a basis for E_λ of eigenvectors of U.

2 From Cayley-Hamilton to the Jordan Canonical Form

2.1 Cayley-Hamilton and a T-invariant decomposition

An extension of the Euclidean Algorithm. Let $f_1(t), f_2(t), \ldots, f_n(t) \in P(F)$ have no nontrivial common factors. Then there exist $s_1(t), s_2(t), \ldots, s_n(t) \in P(F)$ such that

$$1 = \sum_{i=1}^{n} s_i(t) f_i(t).$$

Proof. We will use induction on n.

Base step. If $n < 2$, the statement is vacuous. If $n = 2$, the statement is the Euclidean Algorithm from last time.

Induction step. Suppose $f_1(t), \dots, f_{n-1}(t)$ are all divisible by $g(t)$, and not by any polynomial of higher degree. Then the polynomials

$$\frac{f_1(t)}{g(t)}, \frac{f_2(t)}{g(t)}, \dots, \frac{f_{n-1}(t)}{g(t)}$$

have no common factors, so we can write

$$g(t) = \sum_{i=1}^{n-1} s_i(t) f_i(t)$$

by the induction hypothesis. Since $f_1(t), f_2(t), \dots, f_n(t)$ have no common factors, neither do $g(t)$ and $f_n(t)$, so we can write

$$1 = s(t)g(t) + s_n(t)f_n(t)$$

by the Euclidean Algorithm. Then

$$1 = s_n(t)f_n(t) + \sum_{i=1}^{n-1} \left[s(t)s_i(t)\right] f_i(t)$$

which proves the claim.

Lemma. Let $p(t) \in P(F)$ be irreducible (that is, the only polynomials in $P(F)$ that divide it are nonzero constant multiples of 1 and itself). If $p(t)$ divides the product $f(t)g(t)$, then $p(t)$ either divides $f(t)$ or $g(t)$.

Proof. Suppose $p(t)$ does not divide $f(t)$. Then there exist polynomials $s_1(t), s_2(t) \in P(F)$ such that

$$1 = s_1(t)p(t) + s_2(t)f(t).$$

Then

$$g(t) = s_1(t)p(t)g(t) + s_2(t)f(t)g(t).$$

Since $p(t)$ divides both $s_1(t)p(t)g(t)$ and $s_2(t)f(t)g(t)$, it divides $g(t)$.

We can use induction to extend this result to a larger product.

Lemma. Suppose $f_1(t), f_2(t), \dots, f_n(t) \in P(F)$ are polynomials such that $f_i(t)$ and $f_j(t)$ have no nontrivial common factors whenever $i \neq j$. Let $f(t) = \prod_{i=1}^{n} f(t)$. Then the polynomials

$$\frac{f(t)}{f_1(t)}, \frac{f(t)}{f_2(t)}, \dots, \frac{f(t)}{f_n(t)}$$

have no nontrivial common factors.

Proof. Suppose they do have a nontrivial common factor, and let $p(t)$ be an irreducible factor of that. Then $p(t)$ divides $f(t)$, so by the previous lemma it divides $f_i(t)$ for some i. Without loss of generality, we may assume $p(t)$ divides $f_1(t)$. However, since $p(t)$ divides

$$\frac{f(t)}{f_1(t)} = f_2(t)f_3(t)\cdots f_n(t),$$

it must also divide $f_j(t)$ for some $j \neq 1$, which is a contradiction.

Lemma. Suppose $f(t)$ has no common factors with either $g(t)$ or $h(t)$. Then it also has no common factors with $g(t)h(t)$.

Proof. Let $p(t)$ be a nontrivial irreducible common factor of $f(t)$ and $g(t)h(t)$. Then $p(t)$ divides either $g(t)$ or $h(t)$, which is a contradiction.

Lemma. Let U and T be commuting linear operators on a vector space V. Then $N(U)+N(T) \subseteq N(UT)$.

Proof. Let $u \in N(U)$ and $v \in N(T)$. Then

$$UT(u+v) = UT(u) + UT(v) = TU(u) + UT(v) = 0 + 0 = 0.$$

This result can be extended by induction to a longer list of commuting linear operators.

Theorem. Let T be an operator on a finite-dimensional vector space V whose characteristic polynomial $f(t)$ can be written as

$$f(t) = \prod_{i=1}^{r} f_i(t)$$

where $f_i(t)$ and $f_j(t)$ have no common factors whenever $i \neq j$. Then

$$V = \bigoplus_{i=1}^{r} N(f_i(T)).$$

Proof. Suppose $v \in N(f_j(T)) \cap \left[\sum_{i=1, i\neq j}^{r} N(f_i(T))\right]$ for some j. Then we can write

$$1 = s_1(t) f_j(t) + s_2(t) \prod_{i=1, i\neq j}^{r} f_i(t)$$

so that

$$v = Iv = s_1(T) f_j(T)(v) + \left[s_2(T) \prod_{i=1, i\neq j}^{r} f_i(T) \right](v) = 0 + 0 = 0.$$

Now let $v \in V$. By the above lemmas, we can write

$$1 = \sum_{i=1}^{r} s_i(t) \cdot \frac{f(t)}{f_i(t)}$$

for some $s_1(t), \ldots, s_r(t) \in P(F)$. Then

$$v = \sum_{i=1}^{r} \left(s_i(T) \cdot \frac{f(T)}{f_i(T)} \right)(v)$$

and for each i,

$$f_i(T) \left(s_i(T) \cdot \frac{f(T)}{f_i(T)} \right)(v) = s_i(T) f(T)(v) = 0$$

by Cayley-Hamilton, which proves the claim.

2.2 The Jordan Canonical Form, Part 1

For this section, we will assume that T is a linear operator on a finite-dimensional vector space V (say, of dimension n) whose characteristic polynomial $f(t)$ splits. Then if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T with multiplicities $m_1, m_2, \ldots, m_k$, we have that

$$f(t) = \prod_{i=1}^{k} (\lambda_i - t)^{m_i}.$$

Then the previous theorem tells us that

$$V = \bigoplus_{i=1}^{k} N((T - \lambda_i I)^{m_i}). \tag{1}$$

Definition. For each i, we call $N((T-\lambda_i I)^{m_i})$ the *generalized eigenspace* for the eigenvalue λ_i, and denote it K_{λ_i}.

Theorem. For each i, $\dim K_{\lambda_i} = m_i$.

Proof. It is sufficient to show that $\dim K_{\lambda_i} \leq m_i$ (otherwise the dimensions in (1) can't match up). Since $(T-\lambda_i I)|_{K_{\lambda_i}}$ is nilpotent, its only eigenvalue is 0. Hence the only eigenvalue of $T|_{K_{\lambda_i}}$ is λ_i. Therefore the characteristic polynomial of $T|_{K_{\lambda_i}}$ is equal to

$$(\lambda_i - t)^{\dim K_{\lambda_i}}.$$

Since this must divide $f(t)$, this proves the claim.