MATH 110 Lecture Notes 16

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1 Diagonalization

Exercise 5.2.20. Let $W_1, W_2, \ldots, W_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} W_i = V.$$

Prove that V is the direct sum of $W_1, W_2, \ldots, W_k$ if and only if

$$\dim V = \sum_{i=1}^{k} \dim W_i.$$

Hint: combine bases for each W_i into a single set.

Theorem. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of a linear operator T on a finite-dimensional vector space V. Then

$$\bigoplus_{i=1}^{k} E_{\lambda_i} \subseteq V$$

and equality is achieved if and only if T is diagonalizable.

Proof. Most of this was done last time. Now just count dimensions.

Exercise 5.2.11. Let A be an $n \times n$ matrix which is similar to an upper triangular, and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A, with multiplicities $m_1, \ldots, m_k$. Prove that

$$\mathrm{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$$

and

$$\det(A) = \prod_{i=1}^{k} \lambda_i^{m_i}.$$

2 The Cayley-Hamilton Theorem

Definition. Let T be an operator on a vector space V. Then a subspace W of V is called *T-invariant* if $T(W) \subseteq W$.

Example. Let λ be an eigenvalue of T. Then E_λ is T-invariant.

Definition. The *T-cyclic subspace of V generated by x* is

$$\operatorname{span}\{x, T(x), T^2(x), \dots\}.$$

We can prove that this subspace is T-invariant.

Theorem. Let W be a T-invariant subspace of V. Then the characteristic polynomial of $T|_W : W \to W$ divides that of T.

Proof. Choose a basis β' for W, and expand this to a basis β for V. Then

$$[T]_\beta = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $[T|_W]_{\beta'} = A$.

Theorem. Let T be a linear operator on a finite-dimensional vector space V, and let W be the T-cyclic subspace of V generated by a nonzero vector x. Then

1. for some k, $\{x, T(x), T^2(x), \dots, T^{k-1}(x)\}$ is a basis for W; and
2. if $a_0x + a_1T^1(x) + \cdots + a_{k-1}T^{k-1}(x) + T^k(x) = 0$, then the characteristic polynomial of $T|_W$ is

$$f(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Proof. For (1), suppose $T^k(x)$ is in the span of $\{x, T(x), \dots, T^{k-1}(x)\}$ for some k. Then we can show that $T^m(x)$ is in the same span for all $m \geq k$ by induction on m.

For (2), write down the matrix for $T|_W$ relative to the basis given in (1). Then the characteristic polynomial is the appropriate thing by exercise 19.

Theorem (Cayley-Hamilton). Let T be a linear operator on a finite-dimensional vector space V with characteristic polynomial $f(t)$. Then $f(T) = 0$.

Proof. It is sufficient to show that $f(T)(v) = 0$ for an arbitrary nonzero $v \in V$.

Let W be the T-cyclic subspace generated by v, and let $g(t)$ be the characteristic polynomial of $T|_W$. Then we will show that $g(T)(v) = 0$ and that $g(t)$ divides $f(t)$ by expanding a basis for W to a basis for V. Therefore $f(T)(v) = 0$.

Euclidean Algorithm for Polynomials. Let $f_1(t), f_2(t) \in P(F)$. Then there exists a third polynomial $g(t) \in P(F)$, their greatest common divisor, having the following properties:

1. $g(t)$ divides both $f_1(t)$ and $f_2(t)$.
2. If $h(t)$ divides both $f_1(t)$ and $f_2(t)$, then $h(t)$ also divides $g(t)$.
3. There exist $s_1(t), s_2(t) \in P(F)$ such that $g(t) = s_1(t)f_1(t) + s_2(t)f_2(t)$.

Proof. Part (2) follows from (1) and (3). Without loss of generality, we may assume $\deg f_1(t) \leq \deg f_2(t)$. We will prove (1) and (3) by induction on $\deg f_1(t)$.

Base step. If $f_1(t)$ is a nonzero constant, let $g(t) = 1$, $s_1(t) = [f_1(t)]^{-1}$, and $s_2(t) = 0$. If $f_1(t) = 0$, let $g(t) = f_2(t)$, $s_1(t) = 0$, and $s_2(t) = 1$.

Induction step. There exist polynomials $q(t), r(t) \in P(F)$ such that

$$f_2(t) = q(t)f_1(t) + r(t)$$

and $\deg r(t) < \deg f_1(t)$. By induction, there exist polynomials $g(t), s_1'(t), s_2'(t) \in P(F)$ such that $g(t)$ divides both $r(t)$ and $f_1(t)$, and

$$g(t) = s_1'(t)r(t) + s_2'(t)f_1(t).$$

Then $g(t)$ also divides $f_2(t)$, and

$$g(t) = [s_2'(t) - q(t)s_1'(t)]\, f_1(t) + s_1'(t)f_2(t).$$

Letting $s_1(t) = s_2'(t) - q(t)s_1'(t)$ and $s_2(t) = s_1'(t)$, this proves the claim.

Theorem (application of C-H). Let T be a linear operator on a finite-dimensional vector space V with characteristic polynomial equal to the product $f_1(t)f_2(t)$, where $f_1(t)$ and $f_2(t)$ have no common factors (that is, their GCD is a nonzero constant). Then $V = N(f_1(T)) \oplus N(f_2(T))$.

Proof. By the Euclidean Algorithm, there exist polynomials $s_1(t), s_2(t) \in P(F)$ such that

$$I = s_1(T)f_1(T) + s_2(T)f_2(T).$$

Suppose $v \in N(f_1(T)) \cap N(f_2(T))$. Then

$$v = s_1(T)f_1(T)(v) + s_2(T)f_2(T)(v) = 0 + 0 = 0.$$

Now suppose $v \in V$. We need to show that $v \in N(f_1(T)) + N(f_2(T))$. By Cayley-Hamilton,

$$f_2(T)\,[s_1(T)f_1(T)(v)] = 0 = f_1(T)\,[s_2(T)f_2(T)(v)]\,.$$

Therefore $s_1(T)f_1(T)(v) \in N(f_2(T))$, $s_2(T)f_2(T)(v) \in N(f_1(T))$, and v is equal to the sum of these two vectors, which proves the claim.