MATH 110 Lecture Notes 15

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1 Diagonalization

Theorem. Let T be a linear operator on a vector space V, and let $v_1, v_2, \ldots, v_k$ be eigenvectors with *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof. Suppose

$$\sum_{i=1}^{k} a_i v_i = 0.$$

Then we can apply the operator $\prod_{i=2}^{k} T - \lambda_1 I_V$ to both sides to get

$$0 = \sum_{i=1}^{k} a_i \left(\prod_{i=2}^{k} T - \lambda_1 I_V \right) (v_i) = \sum_{i=1}^{k} a_i \left(\prod_{i=2}^{k} \lambda_i - \lambda_1 \right) v_i = a_1 v_1.$$

Therefore $a_1 = 0$. The result follows by induction on k.

Corollary. Any $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Definition. A polynomial $f(t) \in P(F)$ is said to split over F if $f(t)$ can be expressed as the product of linear and constant factors in $P(F)$.

Examples. The polynomial $t^2 + 1$ splits over $\mathbb{C}$, but not over $\mathbb{R}$. The polynomial $t^2 - 2$ splits over $\mathbb{R}$, but not over $\mathbb{Q}$. (Any polynomial with no roots in a given field does not split over that field.)

Fundamental Theorem of Algebra. Let $f(t) \in P(\mathbb{C})$. Then $f(t)$ splits over $\mathbb{C}$.

Theorem. The characteristic polynomial of any diagonalizable linear operator splits.

Proof. Regard the operator as a matrix. Then the characteristic polynomial of that matrix is the same as that of a diagonal matrix, and the characteristic polynomial of a diagonal matrix can easily be written as a product of linear factors.

Definition. Let λ be an eigenvalue of a linear operator with characteristic polynomial $f(t)$. Then the *multiplicity* of λ is the largest integer m for which $(t - \lambda)^m$ divides $f(t)$. In particular, the multiplicity of any eigenvalue is at least 1.

Example. Find the multiplicities of the eigenvalues of $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$.

Definition. Let T be a linear operator with eigenvalue λ. Then the *eigenspace* of T corresponding to λ, E_λ, is defined to be $N(T - \lambda I)$.

Theorem. Let λ be an eigenvalue of an operator T on a finite dimensional vector space V with multiplicity m. Then $1 \leq \dim E_\lambda \leq m$.

Proof. Choose a basis for E_λ, and expand this to a basis for V. Then the matrix for T relative to this basis has the form

$$\begin{pmatrix} \lambda I_{E_\lambda} & B \\ 0 & C \end{pmatrix}.$$

By a homework exercise, the characteristic polynomial is equal to the product of the characteristic polynomials of λI_{E_λ} and C.