

# MATH 110 Lecture Notes 14

GSI Carter

July 15, 2008

## 1 Eigenvalues and Eigenvectors

Let  $T : V \rightarrow V$  be a linear transformation. Then a vector  $v \in V$  is called an *eigenvector* of  $T$  if  $v \neq 0$  and there exists a scalar  $\lambda$  such that

$$T(v) = \lambda v.$$

The scalar  $\lambda$  is called the *eigenvalue* associated to  $v$ .

To find such vectors  $v$ , we are looking for scalars  $\lambda$  such that

$$T(v) - \lambda v = 0$$

has nonzero solutions, which is true if and only if the linear transformation  $T - \lambda I_V$  has a nonzero null space. Since this is a linear transformation from  $V$  to  $V$ , this is equivalent to  $\det(T - \lambda I)$  being zero. Thus we define the *characteristic polynomial* of  $T$ ,

$$f(t) = \det(T - tI).$$

The eigenvalues of  $T$  are exactly the roots of this polynomial. (You may have seen these definitions for matrices, but they work just as well for linear operators on arbitrary finite dimensional vector spaces.)

**Example.** Find eigenvectors and eigenvalues for  $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ .

**Example.** Find eigenvectors and eigenvalues for  $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ .

**Example.** It is possible for a matrix to have a basis of eigenvectors over  $\mathbb{C}$ , but no eigenvectors over  $\mathbb{R}$ . For example, consider  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ .

**Exercise 5.1.9.** Prove that the eigenvalues of an upper triangular matrix  $M$  are the diagonal entries of  $M$ .

**Exercise 5.1.17(a).** Let  $T$  be the linear operator on  $M_{n \times n}(\mathbb{R})$  defined by  $T(A) = A^t$ . Show that  $\pm 1$  are the only eigenvalues of  $T$ . (Hint: Don't try to compute the characteristic polynomial directly. Which eigenvectors can we come up with easily?)

**Exercise 5.1.21(a).** If  $f(t)$  is the characteristic polynomial of  $A$ , prove that

$$f(t) = (A_{11} - t)(A_{22} - t) \cdots (A_{nn} - t) + q(t),$$

where  $\deg q(t) \leq n - 2$ . Hint: apply mathematical induction to  $n$ .

(b) Show that the trace of  $A$  is equal to  $(-1)^{n-1}$  times the coefficient of  $t^{n-1}$  in  $f(t)$ .

**Exercise 5.1.22(a)** Let  $T$  be a linear operator on a vector space  $V$  over the field  $F$  and let  $g(t)$  be a polynomial with coefficients from  $F$ . Prove that if  $x$  is an eigenvector of  $T$  with corresponding eigenvalue  $\lambda$ , then  $g(T)(x) = g(\lambda)x$ . That is,  $x$  is an eigenvector of  $g(T)$  with corresponding eigenvalue  $g(\lambda)$ . (Use exercise 15, which states that  $T^m(x) = \lambda^m x$  for any  $m > 0$ .)