MATH 110 Lecture Notes 13

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1 Multiplicativity of the Determinant

Theorem. For any $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

Proof. If either A or B is not invertible, then AB is not invertible either. In this case, both $\det(AB)$ and $(\det A)(\det B)$ are 0.

Now suppose A and B are both invertible. Then $A = E_m \cdots E_2 E_1$, where each E_i is an elementary matrix. Then we induct on m.

Corollary. If A is invertible, $\det(A^{-1}) = [\det A]^{-1}$.

2 More Properties

Theorem. For any $A \in M_{n\times n}(F)$, $\det(A^t) = \det A$.

Proof. If A is not invertible, then neither is A^t, and both determinants are zero. If A is invertible, we can write $A = E_m \cdots E_2 E_1$, where each E_i is elementary. Then $A^t = E_1^t E_2^t \cdots E_m^t$, so we must only show that any elementary matrix has the same determinant as its transpose.

Exercise 4.3.12. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called *orthogonal* if $QQ^t = I$. Prove that if Q is orthogonal, then $\det Q = \pm 1$.

Exercise 4.3.23. Given a matrix A, not necessarily square, a $k \times k$ minor of A is the result of choosing k rows of A, k columns of A, and taking the determinant of the resulting submatrix. Prove that the rank of A is the largest value of k for which A has a nonzero $k \times k$ minor.

As a first step, prove that A has full rank if and only if A has a nonzero maximal minor.