MATH 110 Lecture Notes 12

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1 2×2 Determinants

The determinant of a 2×2 matrix is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Theorem. A 2×2 matrix is invertible if and only if its determinant is nonzero.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and suppose $\det A \neq 0$. Then one can verify that

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

by simplying multiplying this matrix and A together.

If $ad - bc = 0$, we can see that A is not invertible by row reduction.

Theorem. For any 2×2 matrix A, $\det A = \det(A^t)$.

Proof. Observe that $ad - bc = ad - cb$.

2 Determinants

Let A be an $n \times n$ matrix. Then the ij-cofactor of A is the scalar

$$c_{ij} = (-1)^{i+j} \det(\tilde{A}_{ij})$$

where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and the j-th column. Then

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}.$$

Example. Show that $\det \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} = 40.$

Theorem. The function $\det : M_{n \times n}(F) \to F$ is linear in each row.

Proof. Use induction on n. For $n = 1$, det is the identity. If $n > 1$, linearity in the first row follows directly from the definition, and linearity in the other rows follows from the induction hypothesis applied to cofactors.

Theorem. Determinants can be evaluated by cofactor expansion along any row. That is, for any i,

$$\det A = \sum_{j=1}^{n} A_{ij} c_{ij}.$$

Proof. See pages 213-215.

Corollary. The determinant of any matrix with two identical rows is 0.

Proof. Use induction on the number of rows. If there are only two rows, see the formula for a 2×2 determinant. If there are three or more rows, choose a row other than the two identical ones and use induction on the cofactors.

Theorem. Interchanging two rows of a square matrix negates its determinant.

Proof. Replace each of the interchanged rows with their sum. Then use linearity in each of the two rows and the above corollary.

Theorem. Adding a multiple of one row to another preserves the determinant.

Proof. Use linearity in the target row and the above corollary.

Exercise 4.2.23. The determinant of an upper triangular matrix is the product of its diagonal entries. We will use induction on the number of rows.

Theorem. A square matrix is invertible if and only if its determinant is nonzero.

Proof. The above theorems tell us that all three types of elementary row operations multiply the determinant of a matrix by some nonzero scalar. Thus the determinant of a matrix is nonzero if and only if the determinant of its reduced row echelon form is nonzero. Since the reduced row echelon form of a square matrix is upper triangular, we can easily compute its determinant. If the original matrix is invertible, the reduced row echelon form is I, and $\det I = 1$. Otherwise, the reduced row echelon form has a row of all zeros, so the determinant is 0.

Exercise 4.2.3. Find the value of k that satisfies the following equation.

$$\det \begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1 + 5c_1 & 3b_2 + 5c_2 & 3b_3 + 5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Exercise 4.2.25. Prove that $\det(kA) = k^n \det A$ for any $A \in M_{n \times n}(F)$.