MATH 110 Lecture Notes 11

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1 Systems of Linear Equations

Consider a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

If we let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

then the above system of equations is equivalent to the matrix equation $Ax = b$.

Suppose $b = 0$. Such a system is called homogeneous, and the set of solutions x is equal to $N(L_A)$. This is a subspace of dimension $n - \text{rank}(A)$. In particular, it is always a nonempty set. If $\text{rank}(A) < n$, this set contains nonzero solutions, for instance if $m < n$.

Suppose now that b is not necessarily zero. Then the solution set of the equation $Ax = b$ is equal to $L_A^{-1}(\{b\})$. For any x and y in this set, $L_A(x - y) = b - b = 0$. Therefore the solution set can be obtained by translating a single solution by arbitrary elements of $N(L_A)$.

Given A and b, a vector x satisfies $Ax = b$ if and only if $PAx = Pb$, where P is any invertible matrix. This fact justifies the Gaussian elimination method of solving system of equations.