MATH 110 Lecture Notes 10

GSI Carter

July 8, 2008

1 Rank

The rank of a matrix is defined to be the rank of its corresponding linear transformation. Thus the rank of A is equal to $\dim CS(A)$.

Theorem. Multiplication on either the left or the right by an invertible matrix preserves rank.

Theorem. Any matrix of rank r can be put in the following form by a series of elementary row and column operations:

$$\begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

Here I_r is the $r \times r$ identity matrix, and each O_i is a zero matrix of some size.

Proof. First put the matrix in reduced row echelon form with row operations. Then put it in the above form with column operations.

Corollary. Taking the transpose of a matrix preserves its rank.

Theorem. Let A and B be matrices for which the product AB is defined. Then $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

Proof. Since $CS(AB) \subseteq CS(A)$,

$$\text{rank}(AB) = \dim CS(AB) \leq \dim CS(A) = \text{rank}(A).$$

Then applying this result to the product $B^t A^t$, we get

$$\text{rank}(AB) = \text{rank}((AB)^t) = \text{rank}(B^t A^t) \leq \text{rank}(B^t) = \text{rank}(B).$$

2 Matrix Inverses

Given an matrix A, if A row reduces to I, then $(A \mid I)$ reduces to $(I \mid A^{-1})$. If A does not row reduce to I, then A is not full rank, so A is not invertible.

Exercise 3.2.5(a). Find the inverse of $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.