MATH 110 Lecture Notes 8

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1 Quotient Spaces

Let $T : V \to W$ be linear. Then the First Isomorphism Theorem states that the induced map $\overline{T} : V/N(T) \to R(T)$ is an isomorphism.

Example 1. Let $V = P_3(\mathbb{R})$, $W = P_2(\mathbb{R})$, and $T : V \to W$ be given by differentiation. Then $W = R(T)$ and $N(T)$ consists of constant polynomials. Then $W \cong V/N(T)$.

Example 2. Let $V = P_3(\mathbb{R})$ and $W = \mathbb{R}$. Then given $a \in \mathbb{R}$, we can define $T : V \to W$ be evaluation at a. That is, $T(f) = f(a)$. It is clear that T is onto, so $W \cong V/N(T)$. We can use this to compute the dimension of $N(T)$.

Rank-Nullity Revisited. Let $T : V \to W$ be linear, with V finite dimensional. Then $R(T) \cong V/N(T)$, so $\dim R(T) = \dim V - \dim N(T)$. This proves the Rank-Nullity Theorem.

Example 3. Let V be finite dimensional, and let $T : V \to V$ be linear. Then $\dim N(T^3) - \dim N(T^2) \leq \dim N(T^2) - \dim N(T)$.

Proof. We must show that $\dim(N(T^3)/N(T^2)) \leq \dim(N(T^2)/N(T))$. Since $T(N(T^3)) \subseteq N(T^2)$, we can define a map

$$\overline{T} : N(T^3) \to N(T^2)/N(T).$$

To find $N(\overline{T})$, we need to find the set of vectors $v \in V$ such that $T(v) \in N(T)$. This set is exactly $N(T^2)$, so there is a one-to-one map

$$\eta : N(T^3)/N(T^2) \to N(T^2)/N(T).$$

Then $N(T^3)/N(T^2) \cong R(\eta) \subseteq N(T^2)/N(T)$. This completes the proof.

2 Change of Coordinate Matrices

We will cover diagonalization later in the course, but since you've all seen it before and it's an important example of change of coordinates, I'm going to diagonalize some matrices (or put them in Jordan normal form).

Example 1. $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$

Example 2. $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

In each case we say that the original matrix is *similar* to its diagonalization (or Jordan normal form).

3 Elementary Matrices

There are three types of elementary row operations:

1. interchanging two rows
2. multiplying a row by a nonzero scalar
3. adding a multiple of a row to some other row

We can show that these operations are performed by multiplication on the left by a square matrix using the following characterization of matrix multiplication:

$$A(\mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_n) = (A\mathbf{b}_1 \mid A\mathbf{b}_2 \mid \cdots \mid A\mathbf{b}_n).$$

There are corresponding columns operations which are performed by multiplication on the right, which we can show using the fact that $(AB)^t = B^t A^t$.