

MATH 110 Lecture Notes 6

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July 1, 2008

1 Composition of Linear Transformations

Theorem. Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear. Then $UT : V \rightarrow Z$ is also linear.

Recall the definition of matrix multiplications. Let $A \in M_{m \times n}(F)$ and $B \in M_{n \times p}(F)$. Then the product AB lies in $M_{m \times p}(F)$, and the (i, j) -th entry of that matrix is given by

$$\sum_{k=1}^n a_{ik} b_{kj}$$

Theorem. Let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear, and let α, β , and γ be ordered bases for V, W , and Z , respectively. Then

$$[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}.$$

Example. Let $U : P_3(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ be defined by $U(f) = f'$ and let $T : P_2(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ be defined by

$$T(f(x)) = \int_0^x f(t) dt.$$

Let $\alpha = (1, x, x^2, x^3)$ and $\beta = (1, x, x^2)$. Then

$$[U]_{\alpha}^{\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

and

$$[T]_{\beta}^{\alpha} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Then $[UT]_{\beta}^{\beta} = [U]_{\alpha}^{\beta} [T]_{\beta}^{\alpha} = I$, which agrees with the Fundamental Theorem of Calculus.

Exercise 2.3.12. Let V , W , and Z be vector spaces, and let $T : V \rightarrow W$ and $U : W \rightarrow Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

(b) Prove that if UT is onto, then U is onto. Must T also be onto?

(c) Prove that if U and T are one-to-one and onto, then UT is also.