MATH 110 Lecture Notes 5

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1 Coordinate Vectors and Matrices for Linear Transformations

Let V be a finite dimensional vector space over a field F, and let $\beta = (v_1, \ldots, v_n)$ be an ordered basis for V. Then for any $x \in V$, there are unique scalars $a_1, \ldots, a_n \in F$ such that $x = \sum_{i=1}^{n} a_i v_i$. Then the vector

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is called the coordinate vector for x relative to the ordered basis β.

Example. Let $V = P_2(\mathbb{R})$. Then $\beta = (1, x, x^2)$ is an ordered basis for V. If $f(x) = 4 + 6x - 7x^2$, then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Now suppose $T : V \to W$ is linear, and let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ be ordered bases for V and W, respectively. Then there are unique scalars a_{ij} such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for all $1 \leq j \leq n$. We call the matrix with (i, j)-th entry equal to a_{ij} the matrix representation of T relative to the ordered bases β and γ, denoted $[T]_\beta^\gamma$. Note that the j-th column of $[T]_\beta^\gamma$ is equal to $[T(v_j)]_\gamma$.

Example. Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation given by $T(f) = f'$. Let $\beta = (1, x, x^2, x^3)$ and $\gamma = (1, x, x^2)$. Then since

$$\begin{aligned} T(1) &= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x^2) &= 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ T(x^3) &= 0 \cdot 1 + 0 \cdot x + 3 \cdot x^3 \end{aligned}$$

we have that

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

2 Spaces of Linear Transformations

Let V and W be vector spaces. Given the usual definition of addition of functions and multiplication of a function by a scalar, it is straightforward to check that $\mathcal{L}(V, W)$, the set of all linear transformations from V to W, is a vector space. If $\dim V = n$ and $\dim W = m$ (with $m, n < \infty$), given ordered bases β and γ, it is straightforward to check that the function from $\mathcal{L}(V, W)$ to $M_{m\times n}(F)$ given by $T \mapsto [T]_\beta^\gamma$ is a linear transformation.

Example. $\mathcal{L}(F^n, F^m) \cong M_{m\times n}(F)$.

Exercise 2.2.15. Let V and W be vector spaces, and let S be a subset of V. Define

$$S^0 = \{T \in \mathcal{L}(V, W) : T(x) = 0 \text{ for all } x \in S\}.$$

Prove the following statements.

(a) S^0 is a subspace of $\mathcal{L}(V, W)$.

(b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

(c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.