MATH 110 Lecture Notes 3

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1 Linear Transformations

Let V and W be vector spaces over a field F. Then a function $T : V \to W$ is called a *linear transformation* if for every $x, y \in V$ and every $\alpha \in F$, we have

- $T(x + y) = T(x) + T(y)$
- $T(\alpha x) = \alpha T(x)$

Such a function can also be referred to as an F-linear transformation to indicate which field F is being used.

Some examples:

- Multiplication by a matrix A.

- Differentiation.

- Integration on a fixed interval $[a, b]$.

Theorem. Let $T : V \to W$ be linear. Then the null space $N(T)$ is a subspace of V, and the range $R(T)$ is a subspace of W.

Theorem. Let $T : V \to W$ be linear. Then T maps any generating set for V to a generating set for $R(T)$.

Theorem. Let $T : V \to W$ be linear. Then

$$\dim N(T) + \dim R(T) = \dim V.$$

We define $\text{rank}(T)$ to be $\dim R(T)$ and $\text{nullity}(T)$ to be $\dim N(T)$.

Theorem. Let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Theorem. A linear transformation is uniquely determined by its behavior on any generating set of its domain.

Exercise 2.1.5. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $T(f(x)) = xf(x) + f'(x)$. Is T linear? If so, is it one-to-one? Onto?

Exercise 2.1.13. Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.