

# MATH 110 Lecture Notes 2

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## 1 Linear Combinations

Given vectors  $v_1, \dots, v_n \in V$ , an expression of the form

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

with  $a_1, \dots, a_n \in F$  is called a linear combination of the vectors  $v_1, \dots, v_n$ . The scalars  $a_1, \dots, a_n$  are referred to as the coefficients of the linear combination.

If  $S$  is a subset of a vector space  $V$ , we denote by  $\text{span}(S)$  the set of all vectors in  $V$  which can be obtained as linear combinations of the elements of  $S$ .

For any subset  $S \subseteq V$ ,  $\text{span}(S)$  is a subspace of  $V$ .

To determine whether one vector lies in the span of some other set of vectors, we can use row reduction.

**Exercise 1.3.3(a).** Does  $(-2, 0, 3)$  lie in the span of  $(1, 3, 0)$  and  $(2, 4, -1)$ ?

**Exercise 1.3.4(a).** Does  $x^3 - 3x + 5$  lie in the span of  $x^3 + 2x^2 - x + 1$  and  $x^3 + 3x - 1$ ?

## 2 Linear Dependence and Independence

Let  $S$  be a subset of a vector space  $V$ . Then we say that  $S$  is linearly independent if, for any nonempty finite subset  $\{v_1, \dots, v_n\} \subseteq S$  and any coefficients  $a_1, \dots, a_n \in F$ , then

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$$

implies that

$$a_1 = a_2 = \dots = a_n = 0.$$

The set  $S$  is said to be linearly dependent if it is not linearly independent. This definition is equivalent to the one from the textbook.

- The empty set is linearly independent.
- Any set  $\{u\}$  for some  $u \neq 0$  is linearly independent.
  
- A set  $\{u, v\}$  is linearly dependent if and only if one of the vectors  $u$  or  $v$  is a scalar multiple of the other.
  
- The standard basis in  $F^n$  is linearly independent.
  
- Let  $V$  be a vector space, and let  $S_1 \subseteq S_2 \subseteq V$ . If  $S_1$  is linearly dependent, then so is  $S_2$ .
  
- If  $S_2$  is linearly independent, then so is  $S_1$ . (This is the contrapositive of the previous statement.)
- The matrices in exercise 1.5.3 are linearly dependent.

### 3 Bases and Dimension

A subset  $\beta$  of a vector space  $V$  is called a *basis* if  $\beta$  is linearly independent and  $\beta$  generates  $V$  (that is,  $V = \text{span}(\beta)$ ).

Some examples:

- The standard basis in  $F^n$ .
  
  
  
  
  
  
  
  
  
  
- The set  $\{1, x, x^2, x^3, \dots\}$  in  $P(F)$ .
  
  
  
  
  
  
  
  
  
  
- Let  $S = \{s_1, \dots, s_n\}$  be a finite set. For each  $i$ , let  $f_i : S \rightarrow F$  be the function sending  $s_i$  to 1 and everything else to 0. Then  $\{f_1, \dots, f_n\}$  is a basis for the vector space of functions from  $S$  to  $F$ . What

if  $S$  is infinite?

- The set of complex numbers  $\mathbb{C}$  is a vector space over  $\mathbb{R}$ , with basis  $\{1, i\}$ .

Some facts about bases:

- Let  $V$  be a vector space with a subset  $\beta$ . Then  $\beta$  is a basis if and only if each element of  $V$  can be written uniquely as a linear combination of the elements of  $\beta$ .

- Any two bases for the same vector space have the same number of elements (or cardinality, in fuller generality).
- Any linearly independent subset of a vector space can be expanded to a basis.
- Any generating set of a vector space contains a subset which is a basis.

**Exercise 1.6.29(a).** Prove that if  $W_1$  and  $W_2$  are finite-dimensional subspaces of a vector space  $V$ , then the subspace  $W_1 + W_2$  is finite-dimensional, and  $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ .

**(b)** Let  $W_1$  and  $W_2$  be finite-dimensional subspaces of a vector space  $V$ , and let  $V = W_1 + W_2$ . Deduce that  $V$  is the direct sum of  $W_1$  and  $W_2$  if and only if  $\dim V = \dim W_1 + \dim W_2$ .