MATH 110 Lecture Notes 1

GSI Carter

June 23, 2008

1 Vector Spaces

A field F is a set with addition and multiplication operations satisfying the following axioms:

- There exists an element $0 \in F$ such that $0 + x = x$ for all $x \in F$.
- For every $x, y, z \in F$, $(x + y) + z = x + (y + z)$.
- For every $x \in F$, there is an element $-x \in F$ such that $x + (-x) = 0$.
- For every $x, y \in F$, $x + y = y + x$.
- For every $x, y, z \in F$, $x(yz) = (xy)z$ and $x(y + z) = xy + xz$.
- There exists an element $1 \in F$ such that $1 \cdot x = x$ for all $x \in F$.
- For every $x, y \in F$, $xy = yx$.
- For every $x \in F$ other than 0, there exists $x^{-1} \in F$ such that $xx^{-1} = 1$.

Some examples are $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

A vector space V over a field F is a set with addition and scalar multiplication operations satisfying the following axioms:

- There exists an element $0 \in V$ such that $0 + v = v$ for all $v \in V$.
- For every $u, v, w \in V$, $(u + v) + w = u + (v + w)$.
- For every $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.
- For every $u, v \in V$, $u + v = v + u$.
- For every $r \in F$ and every $u, v \in V$, $r(u + v) = ru + rv$.
- For every $r, s \in F$ and every $v \in V$, $(r + s)v = rv + sv$.
- For every $r, s \in F$ and every $v \in V$, $(rs)v = r(sv)$.
- For every $v \in V$, $1 \cdot v = v$.

Examples. The following are vector spaces over a field F:

- F^n
- $M_{m\times n}(F)$, the set of $m \times n$ matrices with entries in F
- $P(F)$, the set of polynomials with coefficients in F
- the set of sequences in F

Consequences of the axioms.

- Cancellation law.
- The vector $0 \in V$ is unique.
- For each $x \in V$, $-x$ is unique.
- For any $x \in V$, $0 \cdot x = 0$.
- For any $a \in F$ and any $x \in V$, $(-a)x = -(ax) = a(-x)$.
- For any $a \in F$, $a \cdot 0 = 0$.

Exercise 1.2.13. Let $V = \mathbb{R}^2$, but define addition and scalar multiplication as

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$$
$$c(a_1, a_2) = (ca_1, a_2)$$

Is this a vector space?

Exercise 1.2.18. Let $V = \mathbb{R}^2$, with addition and scalar multiplication defined as

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$
$$c(a_1, a_2) = (ca_1, ca_2)$$

Is this a vector space?

2 Subspaces

Let W be a subset of a vector space V over a field F. Then W is a vector space (with operations inherited from V) if and only if:

- $0 \in W$
- W is closed under scalar multiplication
- W is closed under addition

In this case, we call W a *subspace* of V.

Example. Let A be an $m \times n$ matrix with entries in F. Then $NS(A) = \{x \in F^n : Ax = 0\}$ is a subspace of F^n.

- Since $A \cdot 0 = 0$, $0 \in NS(A)$.
- Let $x, y \in NS(A)$. Then $A(x + y) = Ax + Ay = 0 + 0 = 0$, so $x + y \in NS(A)$.
- Let $c \in F$ and $x \in NS(A)$. Then $A(cx) = c(Ax) = c \cdot 0 = 0$, so $cx \in NS(A)$.

Exercise 1.3.8. Determine whether the following sets are subspaces of $\mathbb{R}^3$.

- $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2, a_3 = -a_2\}$
- $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$
- $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
- $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
- $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
- $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

Exercise 1.3.19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Exercise 1.3.31. Let W be a subspace of a vector space V over a field F. For any $v \in V$ the set $v + W = \{v + w : w \in W\}$ is called the coset of W containing v.

(a) Prove that $v + W$ is a subspace of V if and only if $v \in W$.

(b) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + W : v \in V\}$ of all cosets of W as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

for all $v_1, v_2 \in V$ and

$$a(v + W) = (av) + W$$

for all $v \in V$ and $a \in F$.

(c) Prove that the preceding operations are well-defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$$

and

$$a(v_1 + W) = a(v_1' + W)$$

for all $a \in F$.

(d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the quotient space of V modulo W and is denoted V/W.