The following problems come from past prelim exams.

- (7.1.1) Let p, q, r, and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent? Prove your answer.
  - At 1 each of the polynomials has the value 0.
  - At 0 each of the polynomials has the value 1.
- (7.1.8) Prove that $\{1, (x-a), (x-a)^2, \ldots, (x-a)^n\}$ forms a basis for $P_n(F)$ for any $a \in F$.
- (7.1.9) Let U, V, and W be finite dimensional subspaces of the same vector space. Prove that

  $$\dim U + \dim V + \dim W - \dim(U+V+W) \geq \max\{\dim(U \cap V), \dim(U \cap W), \dim(V \cap W)\}.$$
- (7.1.10) Suppose V is an n-dimensional vector space over the field F. Let $W \subset V$ be a subspace of dimension $r < n$. Show that W is equal to the intersection of all $(n-1)$-dimensional subspaces of V which contain it.
- (7.1.13) Let S be the subspace of $M_{n\times n}$ generated by all matrices of the form $AB - BA$ with $A, B \in M_{n\times n}$. Prove that $\dim S = n^2 - 1$. (Hint: what is the trace of $AB - BA$?)
- (7.1.15) Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^m x = 0$, $T^{m-1}x \neq 0$ for some positive integer m. Show that $x, Tx, T^2x, \ldots, T^{m-1}x$ are linearly independent.
- (7.4.1) Prove that the inverse of an invertible linear transformation is linear.
- (7.4.3) Let V and W be finite dimensional vector spaces, let X be a subspace of W, and let $T : V \to W$ be a linear map. Prove that the dimension of $T^{-1}(X)$ is at least $\dim V - \dim W + \dim X$.