1. Answer the following statements "true" or "false." Throughout this problem, let A and B both be $n \times n$ matrices.
   (a) Let A be similar to B. Then $rk(A) = rk(B)$.
   True.
   (b) Let A be similar to B, and suppose 2 is one of the eigenvalues of A. Then 2 is also one of the eigenvalues of B.
   True. Let $A = P^{-1}BP$. Then

$$A - 2I = P^{-1}BP - 2I = P^{-1}BP - P^{-1}(2I)P = P^{-1}(B - 2I)P.$$

   Then since $A-2I$ does not have full rank, neither does $B-2I$, so 2 must be one of the eigenvalues of B.
   (c) Let 3 be one of the eigenvalues of A. Then 9 is one of the eigenvalues of A^2.
   True.

2. Find matrices S and Λ such that Λ is a diagonal matrix and $A = S\Lambda S^{-1}$.

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -2 & \lambda - 2 \end{vmatrix} = \lambda^2 - 5\lambda + 6 - 2 = (\lambda - 1)(\lambda - 4).$$

The eigenspace for $\lambda = 1$ is

$$NS(A - I) = NS\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\},$$

and the eigenspace for $\lambda = 4$ is

$$NS(A - 4I) = NS\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Therefore we can let

$$S = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$$

and

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$