1. Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find a basis for the following subspace of M_{32}:

$$S = \{A \in M_{32} \mid A\mathbf{u} = \mathbf{0}\}.$$

Let $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$. Then $A \in S$ if and only if

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = A\mathbf{u} = \begin{pmatrix} a-b \\ c-d \\ e-f \end{pmatrix}.$$

Therefore

$$S = \left\{ \begin{pmatrix} a & a \\ c & c \\ e & e \end{pmatrix} : a, c, e \in \mathbb{R} \right\},$$

and a basis for S is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

2. Let

$$B = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$$

and $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find $[\mathbf{x}]_B$.

We are looking for the vector $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, where

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}.$$

This vector equation corresponds to the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & -3 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 0 & 1 & -2 & 3 \\ -1 & 0 & -3 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 & -1 & 4 \\ -1 & 0 & -3 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Therefore $[\mathbf{x}]_B = \begin{pmatrix} 11 \\ -5 \\ -4 \end{pmatrix}$.