MATH 54 Lecture Notes 9

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July 9, 2007

1 Coordinate Vectors

In $\mathbb{R}^2$ or $\mathbb{R}^3$, choosing a different basis to work with is the same as choosing new coordinate axes.

Example 1: the standard basis in $\mathbb{R}^3$. In some textbooks (such as Stewart), the standard basis for $\mathbb{R}^3$ is written $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Every vector in $\mathbb{R}^3$ can be written uniquely as a linearly combination of these vectors. For instance,

$$\begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Given an ordered basis B and a vector $\mathbf{v}$, the coordinate vector for $\mathbf{v}$ with respect to B, written $[\mathbf{v}]_B$, is the vector consisting of the coefficients used to write $\mathbf{v}$ as a linear combination of the elements of B. Thus if $B = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{v} = [\mathbf{v}]_B$.

Example 2: a plane in $\mathbb{R}^3$. Consider the plane $x+y+z=0$ in $\mathbb{R}^3$. Suppose we want to know how far away certain points are from this plane. We can solve this problem by choosing new coordinate axes so that the plane becomes the xy-plane. Then all we have to do is look at the z-coordinates of the points in question.

The first two coordinate axes must lie inside the plane. Let $\mathbf{v}_1 = (1, -1, 0)^T$ and $\mathbf{v}_2 = (0, 1, -1)^T$. Then let $\mathbf{v}_3 = \frac{1}{\sqrt{3}}(1, 1, 1)^T$. (This vector is perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$ and of length 1. The techniques used to come up with such a vector will be covered after the midterm.) Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a new basis for $\mathbb{R}^3$, which means every vector in $\mathbb{R}^3$ can be written uniquely as a linearly combination of these three vectors, and the coefficients used to do so will form coordinate vectors as above.

Let $\mathbf{x} = (1, 0, 1)$. In order to figure out the coordinates for $\mathbf{x}$ with respect to our new coordinate axes, we need to find coefficients a_1, a_2, and a_3 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{x}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, there will be a unique solution for the unknowns a_1, a_2, and a_3. This vector equation corresponds to the augmented matrix

$$\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} & 1 \\ -1 & 1 & \frac{1}{\sqrt{3}} & 0 \\ 0 & -1 & \frac{1}{\sqrt{3}} & 1 \end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} & 1 \\ 0 & 1 & -\frac{1}{\sqrt{3}} & -1 \\ 0 & 0 & \sqrt{3} & 2 \end{pmatrix}.$$

Thus $a_3 = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$, $a_2 = -1 + \frac{a_3}{\sqrt{3}} = -\frac{1}{3}$, and $a_1 = 1 - \frac{a_3}{\sqrt{3}} = \frac{1}{3}$. Thus, the coordinate vector for $\mathbf{x}$ with respect to our new basis is $\left(\frac{1}{3}. -\frac{1}{3}, \frac{2\sqrt{3}}{3}\right)$. Therefore $\mathbf{x}$ lies at a distance of $\frac{2\sqrt{3}}{3}$ from the plane $x + y + z = 0$.

Example 3: interpolation polynomials. Suppose we want to be able to easily give the polynomial in P_2 which takes given values at the points $x = -1, 0, 1$. We can do this by first choosing an appropriate basis for P_2. This is like choosing new "axes" in P_2 so that the values at -1, 0, or 1 are simply the three different coordinates.

The first element of our basis will be a polynomial that takes the value 1 at $x = -1$ and vanishes at 0 and 1. Let

$$p_1 = \frac{x(x-1)}{(-1)(-1-1)} = \frac{x^2 - x}{2}.$$

Then we want a polynomial that takes the value 1 at $x = 0$ and vanishes at -1 and 1. Let

$$p_2 = \frac{(x+1)(x-1)}{(1)(-1)} = 1 - x^2.$$

Now we want a polynomial that takes the value 1 at $x = 1$ and vanishes at 0 and -1. Let

$$p_3 = \frac{x(x+1)}{(1)(1+1)} = \frac{x^2 + x}{2}.$$

Then $\{p_1, p_2, p_3\}$ is a basis for P_2. Since we already know that $\dim P_2 = 3$, it suffices to check that this set is linearly independent *or* that it spans P_2. See below.

Let $B = \{p_1, p_2, p_3\}$. Then if q is the parabola which passes through the points $(-1, 3)$, $(0, -1)$, and $(1, 2)$, then

$$[q]_B = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

so that

$$q = 3p_1 - p_2 + 2p_3 = \frac{3x^2 - 3x - 2 + 2x^2 + 2x^2 + 2x}{2} = \frac{7x^2 - x - 2}{2}.$$

2 Bases

Fact 1. If we start with a linearly independent set of vectors inside a vector space V, we can add more vectors to it until it becomes a basis.

Fact 2. If we start with a set of vectors inside V which spans V, we can remove vectors from it until it becomes a basis.

Fact 3. Any two bases for the same vector space V must have the same number of elements.

Fact 4. In order to check whether a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ inside V is a basis for V, it is enough to check any *two* of the following:

- $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.
- $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
- $\dim V = n$.

This can be proven from the above facts.