MATH 54 Lecture Notes 8

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July 6, 2007

1 Problems

Spans inside M_{22}. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{aligned} A^2 &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} \\ &= \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix} - \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= (a+d)A - (ad-bc)I. \end{aligned}$$

Therefore $A^2 \in \text{Span}\{A, I\}$, and the set $\{A^2, A, I\}$ is linearly dependent. We can prove by induction that $\text{Span}\{I, A\} = \text{Span}\{I, A, A^2, \ldots, A^k\}$ for all $k \geq 1$. The base step, $k = 1$, is trivial. To prove the induction step, we must establish the equality

$$\text{Span}\{I, A, A^2, \ldots, A^{k-1}\} = \text{Span}\{I, A, A^2, \ldots, A^k\}$$

for all $k \geq 2$. It is clear that $\text{Span}\{I, A, A^2, \ldots, A^{k-1}\} \subseteq \text{Span}\{I, A, A^2, \ldots, A^k\}$. To show that $\text{Span}\{I, A, A^2, \ldots, A^k\} \subseteq \text{Span}\{I, A, A^2, \ldots, A^{k-1}\}$, it suffices to show that $A^k \in \text{Span}\{I, A, A^2, \ldots, A^{k-1}\}$. This follows from the equality

$$A^k = (a+d)A^{k-1} - A^{k-2}.$$

We have now established that $A^n \in \text{Span}\{A, I\}$ for all $n \geq 0$. If A is invertible, applying this result to A^{-1} shows that $A^{-n} \in \text{Span}\{A^{-1}, I\}$ for all $n \geq 0$. Since A is invertible, $ad - bc \neq 0$, so

$$I = \frac{a+d}{ad-bc} \cdot A - \frac{1}{ad-bc} \cdot A^2$$

and hence

$$A^{-1} = \frac{a+d}{ad-bc} \cdot I - \frac{1}{ad-bc} \cdot A.$$

Then since $A^{-1} \in \text{Span}\{A, I\}$, $\text{Span}\{A^{-1}, I\} \subseteq \text{Span}\{A, I\}$, so that $A^n \in \text{Span}\{A, I\}$ for all $n \in \mathbb{Z}$.

Basis inside P_3. Let $S = \{p \in P_3 \mid p(1) = 0\}$. If we wish to find a basis for S, first we should try to write down a more suitable description of a general element of S. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ (i.e. an arbitrary element of P_3). Then $f \in S$ if and only if

$$0 = f(1) = a_0 + a_1 + a_2 + a_3.$$

We can use this equation to eliminate a_3 and write down the following description of S:

$$S = \{a_0 + a_1x + a_2x^2 - (a_0 + a_1 + a_2)x^3 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

Then we get the following basis for S:

$$\{1 - x^3, x - x^3, x^2 - x^3\}.$$

Note that the basis is a set of three polynomials (not three vectors in $\mathbb{R}^4$, not infinitely many polynomials).

2 Null Spaces and Row Spaces

Let A be an $m \times n$ matrix with rows $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. The *row space* of A, denoted $RS(A)$, is the subspace of $\mathbb{R}^n$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. To see both why this space is interesting and how to compute it, let's look at what the three different kinds of elementary row operations do to the row space of a matrix.

Swapping two rows. Clearly no effect. Since vector addition is commutative, the order in which a set of vectors is listed has no bearing on their span.

Multiplying a row by a nonzero scalar. Also no effect. Multiplying a vector by a nonzero scalar does not affect its span.

Adding a multiple of one row to another. Suppose we were to add c times row i to row j. Then we want to compare

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j, \ldots, \mathbf{v}_m\}$$

with

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_j + c\mathbf{v}_i, \ldots, \mathbf{v}_m\}.$$

Since $\mathbf{v}_j \in \text{Span}\{\mathbf{v}_j + c\mathbf{v}_i, \mathbf{v}_i\}$ and $\mathbf{v}_j + c\mathbf{v}_i \in \text{Span}\{\mathbf{v}_j, \mathbf{v}_i\}$, the two above spans are equal. Therefore this row operation also has no effect on the row space of a matrix.

Now let E be some arbitrary $m \times m$ invertible matrix. We wish to compare $RS(A)$ with $RS(EA)$. Since E is invertible, we can write

$$E = E_1E_2 \cdots E_k,$$

where each E_i is an elementary matrix. Since the elementary row operations leave the row space of a matrix unchanged, so does multiplication on the left by an elementary matrix. Therefore

$$RS(A) = RS(E_k A) = RS(E_{k-1} E_k A) = \cdots = RS(EA).$$

Multiplication on the left by an invertible matrix, or any arbitrary sequence of elementary row operations, preserve the row space of a matrix. Therefore, if we want a basis for $RS(A)$, we can row reduce A first, and then simply list all the nonzero rows of the result.

Now consider $NS(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \mathbf{0}\}$, and again let E be some arbitrary $m \times m$ invertible matrix. Then $EA\mathbf{v} = \mathbf{0}$ if and only if $A\mathbf{v} = \mathbf{0}$ by theorem 1.50. Therefore $NS(EA) = NS(A)$. Both the null space and row space of any matrix are *preserved* by multiplication on the left by invertible matrices, or by arbitrary sequences of elementary row operations. Note that this does not work with multiplication on the right by an invertible matrix. That is, it could be the case that $RS(A) \neq RS(AF)$ or $NS(A) \neq NS(AF)$ if F is an $n \times n$ invertible matrix.

In the case of a matrix in row echelon form, the dimension of the row space will be equal to the number of pivots, and the dimension of the null space will be equal to the number of free variables or columns without pivots. Therefore,

$$\dim RS(A) + \dim NS(A) = n,$$

where n is the number of columns of A.

Example. Compute bases for the row space and null space of

$$A = \begin{pmatrix} 1 & -1 & 1+i \\ -2 & 0 & -2-2i \\ -2i & 1 & 2-2i \end{pmatrix}.$$

3 Column Spaces and Rank

Let A again be an $m \times n$ matrix. Then the column space of A, denoted $CS(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns of A. Some properties of this space:

- $CS(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$. This is another way of saying that the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.
- $CS(A) = RS(A^T)$. The columns of A are the same as the rows of A^T.
- Let U be a row echelon matrix obtained by row reducing A. Then a basis of $CS(U)$ is given by the set of all columns of U containing pivots, which can be seen by building up a basis one vector at a time. Also, a basis

for $CS(A)$ is given by the set of all columns *in the original matrix* A which have pivots in U. The proof of this is somewhat complicated and can be found in the textbook. That proof is based largely on the fact that $NS(A) = NS(U)$. This also implies that $\dim CS(A)$ is equal to the number of pivots of U, and therefore $\dim CS(A) = \dim RS(A)$.

- If F is an invertible $n \times n$ matrix, then $CS(A) = CS(AF)$, since F^T is also invertible and hence

$$CS(A) = RS(A^T) = RS(F^T A^T) = RS((AF)^T) = CS(AF).$$

Since $\dim CS(A) = \dim RS(A)$, we call the *rank* of A, or $rk(A)$, this common quantity. Thus we arrive at the Rank-Nullity Theorem:

$$rk(A) + \dim NS(A) = n$$

for any $m \times n$ matrix A.

Example. Compute a basis for the column space of

$$A = \begin{pmatrix} 1 & -1 & 1+i \\ -2 & 0 & -2-2i \\ -2i & 1 & 2-2i \end{pmatrix}.$$