MATH 54 Lecture Notes 7

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1 Linear Independence

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are said to be *linearly independent* if

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_r\mathbf{v}_r = \mathbf{0}$$

implies that

$$a_1 = a_2 = \cdots = a_r = 0.$$

A set of vectors is said to be *linearly dependent* if they are not linearly independent.

Examples:

- $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are linearly independent.
- Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{u} = (1, 1, 0)$. Then these three vectors are linearly dependent, since $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{u} = \mathbf{0}$.

In $\mathbb{R}^n$, any set of vectors with more than n elements is linearly dependent over $\mathbb{R}$. In $\mathbb{C}^n$, any set of vectors with more than n elements is linearly dependent over $\mathbb{C}$.

Exercise 1. Since $3\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{0}$, they are linearly dependent.

Exercise 4. Since $2p_1 + p_2 = 0$, they are linearly dependent.

Exercise 6. Since $f_1(x) + f_2(x) - f_3(x) = 0$ for all x, they are linearly dependent.

Exercise 8. Any dependence relation corresponds to a solution to $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Since A is invertible, there is only the trivial solution, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. If there had been other solutions, $\mathbf{v}_1$ and $\mathbf{v}_2$ would be linearly dependent.

Exercise 28. Without loss of generality, suppose the subset under consideration is $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$. Then suppose $\sum_{i=1}^{k} a_i \mathbf{u}_i = \mathbf{0}$. Then

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_k\mathbf{u}_k + 0 \cdot \mathbf{u}_{k+1} + 0 \cdot \mathbf{u}_{k+2} + \cdots + 0 \cdot \mathbf{u}_n = \mathbf{0}.$$

Since the set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is linearly independent, this implies that $a_1 = a_2 = \cdots = a_k = 0$. Therefore the set $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is also linearly independent.

Exercise 30. This is the contrapositive of exercise 28.

Exercise 34. Suppose $\sum_{i=1}^{k+1} a_i \mathbf{v}_i = \mathbf{0}$. Assume $a_{k+1} \neq 0$. Then we can write

$$\mathbf{v}_{k+1} = -\frac{\sum_{i=1}^{k} a_i \mathbf{v}_i}{a_{k+1}} = \sum_{i=1}^{k} -\frac{a_i}{a_{k+1}} \cdot \mathbf{v}_i$$

which implies that $\mathbf{v}_{k+1} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, a contradiction. Therefore our assumption is false, so $a_{k+1} = 0$. Thus $\sum_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}$. Since the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent, we have that

$$a_1 = a_2 = \cdots = a_k = a_{k+1} = 0.$$

Therefore the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is also linearly independent.

2 Bases

Let V be a vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r \in V$. Then we say that these vectors are a *basis* for V if

- they are linearly independent, and
- they span V (i.e. $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$).

The vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a basis for $\mathbb{R}^3$ (or $\mathbb{C}^3$). They are called the *standard basis.*

Exercise 10. For any $\mathbf{b} \in \mathbb{R}^2$, we want to know whether there exist a_1 and a_2 such that

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 = \mathbf{b}.$$

Equivalently, we can ask whether there is a solution to the matrix equation

$$\begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathbf{b}.$$

There is a unique solution for any $\mathbf{b}$, since the matrix is invertible. This tells us that $\mathbf{x}_1$ and $\mathbf{x}_2$ span $\mathbb{R}^2$. The uniqueness of the solution in the case $\mathbf{b} = \mathbf{0}$ tells us that they are linearly independent. Therefore they are a basis for $\mathbb{R}^2$.